УДК 517.9

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CAUCHY PROBLEMS AND INVARIANT MEASURES FOR ONE STOCHASTIC FUNCTIONAL-DIFFERENTIAL EQUATION

We deal with Cauchy problem for one stochastic functional-differential equation. We study the existence, uniqueness and continuous dependence on initial function of so-called mild solution to this problem. We have also obtained its Markovian and Feller property and obtained sufficient conditions of invariant measure existence in terms of coefficients.

Розглянута задача Коші для стохастичного функціонально-диференціального рівняння. Досліджено існування, єдиність та неперервну залежність від початкових даних м'якого розв'язку цієї задачі. Також встановлено властивість марковості та феллеровості та отримано достатні умови існування інваріантної міри у термінах коефіцієнтів задачі.

1. Introduction. In the given paper we study the following Cauchy problem

$$du(t,x) = \big(\Delta_x u(t,x) + f(u_t(x))\big)dt + \sigma(u_t(x))dW(t,x),\ 0 < t \le T,\ x \in \mathbb{R}^d, \quad (1)$$

$$u(t,x) = \phi(t,x),\ -h \le t \le 0,\ x \in \mathbb{R}^d,\ h > 0, \quad (1^*)$$

where $\Delta_x \equiv \sum_{i=1}^{d} \partial^2_{x_i}$ is d-measurable Laplace operator, $\partial^2_{x_i} \equiv \frac{\partial^2}{\partial x_i^2}$, $i \in \{1,\dots,d\}$, $W(t,x)$, $x \in \mathbb{R}^d$, is $L_2(\mathbb{R}^d)$-valued Q-Wiener process, f and σ are some given functionals to be specified later, ϕ is an initial datum function, and $u_t(x) = u(t+\theta,x)$, $-h \le \theta \le 0$, $x \in \mathbb{R}^d$.

Many authors have been dealing with such a problem in bounded domains with abstract elliptic operators. As particular result they have considered such problems with Laplacian in bounded domains. Let note that we consider problem (1)–(1*) in an unbounded domain. The principal difference of this problem from the problem in a bounded domain is that a semigroup $\{S(t), t \ge 0\}$, generated by the Laplace operator in a bounded domain, possesses the exponential contraction property. Since we have been dealing with Laplacian in the whole space, where an exponential estimate is not valid, our results do not follow from abstract results, obtained earlier by others.

This paper is organised as follows. Firstly, we introduce a statement of the problem and formulate our main results. Then we introduce preliminary facts and necessary notions, needed in what follows. Next sections are devoted to the proof.

2. Mathematical formulation of the problem and main results. This section is supposed to be a gentle introduction to our problem with as little theory as possible.

Throughout the paper we assume that all random objects are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Henceforth $L_2(\mathbb{R}^d)$ will note Hilbert space with the norm $\|g(\cdot)\|_{L_2(\mathbb{R}^d)} = \sqrt{\int_{\mathbb{R}^d} g^2(x)dx}$. Let $\{e_n(x), n \in \{1,2,\dots\}\}$ be an orthonormal basis on $L_2(\mathbb{R}^d)$, which is uniformly bounded, i.e., $\sup_{n\in\{1,2,\dots\}} ess\sup_{x\in\mathbb{R}^d} |e_n(x)| \le 1$.

Let Q be a nonnegative operator. We now define $L_2(\mathbb{R}^d)$-valued Q-Wiener process $W(t,x) = W(t,\cdot)$, $t \geq 0$, $x \in \mathbb{R}^d$, as follows $W(t,\cdot) = \sum\limits_{n=1}^{\infty} \sqrt{\lambda_n} e_n(\cdot)\beta_n(t)$, $t \geq 0$, where $\{\beta_n(t), n \in \{1,2,\dots\}\} \subset \mathbb{R}$ are mutually independent standard real-valued one-dimensional Brownian motions on $t \geq 0$, $\{\lambda_n, n \in \{1,2,\dots\}\}$ is a sequence of positive real numbers, satisfying $TrQ = \sum\limits_{n=1}^{\infty} \lambda_n < \infty$. Let $\{\mathcal{F}_t, t \geq 0\}$ be a normal filtration on $\mathcal{F}$. We assume that $W(t,\cdot)$, $t \geq 0$, is a Q-Wiener process with respect to a filtration $\{\mathcal{F}_t, t \geq 0\}$, i.e., $W(t,\cdot)$, $t \geq 0$, is $\mathcal{F}_t$-measurable, and the increments $W(t+h_0,\cdot) - W(t,\cdot)$ are independent of $\mathcal{F}_t$ for all $h_0 > 0$ and $t \geq 0$. Let denote by $H_0^\rho = L_2^\rho(\mathbb{R}^d)$ real Hilbert space with the norm $\|g(\cdot)\|_{H_0^\rho} = \sqrt{\int\limits_{\mathbb{R}^d} g^2(x)\rho(x)dx}$, by $H_1^\rho = L_2([-h;0]; H_0^\rho)$ — real Hilbert space with the norm $\|h(\cdot)\|_{H_1^\rho} = \sqrt{\int\limits_{-h}^{0}\int\limits_{\mathbb{R}^d} h^2(\theta,x)\rho(x)dxd\theta}$. Let $H^\rho = H_0^\rho \times H_1^\rho$ denote Hilbert space of vectors $U(t,x) = \begin{pmatrix} u(t,x) \in H_0^\rho,\ t \geq 0,\ x \in \mathbb{R}^d, \\ u_t(x) = u(t+\theta,x) \in H_1^\rho,\ -h \leq \theta \leq 0,\ x \in \mathbb{R}^d \end{pmatrix}$, $\Phi(t,x) = $ $= \begin{pmatrix} \phi(0,x) \in H_0^\rho,\ x \in \mathbb{R}^d, \\ \phi(t,x) \in H_1^\rho,\ -h \leq t \leq 0,\ x \in \mathbb{R}^d \end{pmatrix}$ with the norm

$$\|U(t,\cdot)\|_{H^\rho} = sqrt\|u(t,\cdot)\|^2_{H_0^\rho} + \|u_t(\cdot)\|^2_{H_1^\rho},$$

$$\|\Phi(t,\cdot)\|_{H^\rho} = \sqrt{\|\phi(0,\cdot)\|^2_{H_0^\rho} + \|\phi(t,\cdot)\|^2_{H_1^\rho}}.$$

We impose the following two conditions.

1) $\{f,\sigma\}\colon H_1^\rho \to H_0^\rho$ are such that for some constant $L > 0$

$$\|f(u)\|_{H_0^\rho} + \|\sigma(u)\|_{H_0^\rho} \leq L\big(1 + \|u\|_{H_1^\rho}\big),\ u \in H_1^\rho,$$
$$\|f(u) - f(v)\|_{H_0^\rho} + \|\sigma(u) - \sigma(v)\|_{H_0^\rho} \leq L\|u - v\|_{H_1^\rho},\ \{u,v\} \subset H_1^\rho.$$

2) The initial-datum function $\phi\colon H_0^\rho \to H_0^\rho$ is $\mathcal{F}_0$-measurable random function, independent of $W(t,x)$, $t \geq 0$, $x \in \mathbb{R}^d$, and such that

$$\mathbf{E}\int\limits_{-h}^{0} \|\phi(t,\cdot)\|^p_{H_0^\rho} dt < \infty,\ \mathbf{E}\|\phi(0,\cdot)\|^p_{H_0^\rho} < \infty,\ p \geq 2.$$

Let $S(t)\colon H_0^\rho \to H_0^\rho$. From now on we use the notation $S(t)g(\cdot)$, $g \in H_0^\rho$, to denote the convolution $\big(S(t)g(\cdot)\big)(x) = \int\limits_{\mathbb{R}^d} \mathscr{K}(t, x-\xi)g(\xi)d\xi$, $x \in \mathbb{R}^d$, $g \in H_0^\rho$. It is known from semigroup theory that by this rule operators $\{S(t), t \geq 0\}$ generate solution

$$u(t,x) = \big(S(t)g(\cdot)\big)(x) = \int\limits_{\mathbb{R}^d} \mathscr{K}(t, x-\xi)g(\xi)d\xi \qquad (2)$$

of the problem

$$\partial_t u(t,x) = \Delta_x u(t,x),\ t>0,\ x\in\mathbb{R}^d,$$
$$u(0,x) = g(x),\ x\in\mathbb{R}^d,$$

and besides these operators create (C_0)-semigroup of operators, an infinitesimal generator of which is the Laplace operator Δ_x. Here

$$\mathscr{K}(t,x) = \begin{cases} \frac{1}{(4\pi t)^{\frac{d}{2}}}\exp\Big\{-\frac{|x|^2}{4t}\Big\}, & t>0,\quad x\in\mathbb{R}^d,\\ 0, & t<0,\quad x\in\mathbb{R}^d,\end{cases}$$

denotes the source function (diffusion kernel) of the heat equation.

We next proceed with a rigorous definition of mild solution of (1) — (1*).

Definition 1. *A continuous random process $u\colon H_0^\rho \to H_0^\rho$ is called a* ***mild solution (solution)*** *to* (1) — (1*) *on $0\le t\le T$ provided*

1) *It is $\mathcal{F}_t$-measurable for $0\le t\le T$.*

2) *It satisfies the integral equation*

$$u(t,x) = \int\limits_{\mathbb{R}^d}\mathscr{K}(t,x-\xi)\phi(0,\xi)d\xi + \int\limits_0^t\int\limits_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)f(u_s(\xi))d\xi ds$$
$$+\int\limits_0^t\sum_{n=1}^{\infty}\sqrt{\lambda_n}\Bigg(\int\limits_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)\sigma(u_s(\xi))e_n(\xi)d\xi\Bigg)d\beta_n(s),$$
$$0<t\le T,\ x\in\mathbb{R}^d, \tag{3}$$

$$u(t,x) = \phi(t,x),\ -h\le t\le 0,\ x\in\mathbb{R}^d,\ h>0, \tag{3*}$$

$$u(0,x) = \phi(0,x),\ x\in\mathbb{R}^d. \tag{3**}$$

3) *It satisfies the condition* $\mathbf{E}\int\limits_0^T\|u(t,\cdot)\|^p_{H_0^\rho}dt<\infty,\ p\ge 2.$

Remark 1. *It is assumed in the definition above that all the integrals from* (3) *are well defined.*

The following theorem is true for such a solution.

Theorem 1 (existence and uniqueness)**.** *Let conditions from* 1), 2) *be true. Then there exists a unique solution $u(t,\cdot)$ to* (1) — (1*) *for $0\le t\le T$.*

From the theorem above the following result follows.

Corollary 1. *Suppose conditions of theorem 1 are valid. Then for* $U(t,x) = \begin{pmatrix} u(t,x)\in H_0^\rho,\ t\ge 0,\ x\in\mathbb{R}^d,\\ u_t(x) = u(t+\theta,x)\in H_1^\rho,\ -h\le\theta\le 0,\ x\in\mathbb{R}^d\end{pmatrix}$, *where $u(t,\cdot)$ is the solution* *to* (1) — (1*) *for* $0 \le t \le T$*, and* $u_t(\cdot)$ *solves the problem*

$$u_t(x) = \int\limits_{\mathbb{R}^d} \mathscr{K}(t+\theta, x-\xi)\phi(0,\xi)d\xi + \int\limits_0^{t+\theta}\int\limits_{\mathbb{R}^d} \mathscr{K}(t+\theta-s, x-\xi) f(u_s(\xi))d\xi ds$$

$$+ \int\limits_0^{t+\theta} \sum_{n=1}^{\infty} \sqrt{\lambda_n}\Big(\int\limits_{\mathbb{R}^d} \mathscr{K}(t+\theta-s, x-\xi)\sigma(u_s(\xi))e_n(\xi)d\xi\Big) d\beta_n(s),$$

$$-\theta \le t \le T-\theta,\ x \in \mathbb{R}^d, \tag{4}$$

$$u_t(x) = \phi_t(x),\ -h-\theta \le t \le -\theta,\ x \in \mathbb{R}^d,\ h > 0, \tag{4*}$$

$$u(0,x) = \phi(0,x),\ x \in \mathbb{R}^d, \tag{4**}$$

we have $\mathbf{E}\|U(t,\cdot)\|^p_{H^\rho} < \infty$.

Next we are interested whether the unique solution depends continuously on the initial condition. At first let us note that if we replace the initial range $[-h, 0]$ from (1*) by $[s-h, s]$ for arbitrary $0 \le s \le t$, it will be possible to guarantee existence and uniqueness of the solution to (1) — (1*) for $0 \le s \le t$ with $\mathfrak{F}_s$-measurable initial datum function $\phi(t,\cdot)$, satisfying assumptions from 2) for $s-h \le t \le s$. Such a solution will be temporarily denoted by $u(t,s,\cdot,\phi)$. Consequently, if we define $\mathfrak{F}_s$-measurable initial datum function $\phi(s+\theta,\cdot,\omega) \in H_1^\rho$, $\theta \in [-h, 0]$, satisfying conditions from assumption 2), then $u(s+\theta, s,\cdot,\phi) = \phi(s+\theta,\cdot)$ and $u(t,s,\cdot,\phi)$ satisfies (3) for $t \ge s$. Let denote by $u_t(s,\cdot,\phi) = u(t+\theta,s,\cdot,\phi)$, $-h \le \theta \le 0$, shift of solution u such that $u_s(s,\cdot,\phi) = u(s+\theta,s,\cdot,\phi) = \phi(s+\theta,\cdot)$.

Let $\mathfrak{D}_1$ be σ-algebra of Borel subsets from H_1^ρ. If for any set $A_1 \in \mathfrak{D}_1$ we define

$$\mu_t(A_1) = \mathbf{P}\{u_t(s,\cdot,\varphi) \in A_1\} = P(s,\varphi,t,A_1), \tag{5}$$

then $u_t(s,\cdot,\varphi)$ defines a measure on $\mathfrak{D}_1$. Function (5) is said to be a ***transition function***, corresponding to the random process $u_t(s,\cdot,\varphi)$. Similarly to the finite-dimensional case from [2], it is possible to show that this function possess all standard properties of the transition probability. The following theorem is valid.

Theorem 2 (the Markovian property)**.** *Under assumptions of theorem* 1 *the process* $u_t(s,\cdot,\phi)$*, where* ϕ *satisfies conditions from* 2)*, is the Markov process on* H_1^ρ *with the transition function* μ_t*, defined by* (5)*.*

From this theorem the following result follows.

Corollary 2. *Suppose conditions of existence theorem* 1 *are valid. Then* $U(t,x,\phi) = \begin{pmatrix} u(t,s,x,\phi) \in H_0^\rho,\ t \ge s,\ x \in \mathbb{R}^d, \\ u_t(s,x,\phi) = u(t+\theta,s,x,\phi) \in H_1^\rho,\ -h \le \theta \le 0,\ x \in \mathbb{R}^d, \end{pmatrix}$ *is the Markov process on* H^ρ *with the transition function* $\mu_t(A) = \mathbf{P}\{U(t,\cdot,\phi) \in A\}$*,* $A \in \mathfrak{D}$*, where* $\mathfrak{D}$ *is* σ*-algebra of Borel subsets from* H^ρ*.*

Let denote by $\mathfrak{B}_b(H_1^\rho)$ the Banach space of all real bounded Borel functions, defined on H_1^ρ, by $\mathfrak{C}_b(H_1^\rho)$ — the Banach space of all real bounded continuous functions, defined on H_1^ρ. If operator $P_{s,t}f(\varphi) = \mathbf{E}f(u_t(s,\cdot,\varphi))$, $0 \le s \le t \le T$,

$\varphi \in H_1^\rho$, is bounded and continuous for any $f \in \mathfrak{C}_b(H_1^\rho)$, then it will be said that $P_{s,t}$ possess the Feller property. Hence, if operator

$$P_{s,t}g(\varphi) = \mathbf{E}g(U(t,\,\cdot\,,\varphi)),\ 0 \le s \le t \le T,\ \varphi \in H_1^\rho, \tag{6}$$

where $U(t,x,\varphi) = \begin{pmatrix} u(t,s,x,\varphi) \in H_0^\rho,\ t \ge s,\ x \in \mathbb{R}^d, \\ u_t(s,x,\varphi) = u(t+\theta,s,x,\varphi) \in H_1^\rho,\ -h \le \theta \le 0,\ x \in \mathbb{R}^d \end{pmatrix}$, is bounded and continuous for any $g \in \mathfrak{C}_b(H^\rho)$, then it possess the Feller property.

Theorem 3 (continuous dependence on initial datum and the Feller property)**.** *Under assumptions of theorem* 1 *there exists* $C(T) > 0$ *such that for* $\phi(t,\,\cdot\,) \in H_1^\rho$, $\phi(0,\,\cdot\,) \in H_0^\rho$, $\phi_1(t,\,\cdot\,) \in H_1^\rho$, $\phi_1(0,\,\cdot\,) \in H_0^\rho$ *the next processes*

$$U(t,x,\phi) = \begin{pmatrix} u(t,x,\phi) \in H_0^\rho,\ t \ge 0,\ x \in \mathbb{R}^d, \\ u_t(x,\phi) = u(t+\theta,x,\phi) \in H_1^\rho,\ -h \le \theta \le 0,\ x \in \mathbb{R}^d \end{pmatrix}$$

and

$$U(t,x,\phi_1) = \begin{pmatrix} u(t,x,\phi_1) \in H_0^\rho,\ t \ge 0,\ x \in \mathbb{R}^d, \\ u_t(x,\phi_1) = u(t+\theta,x,\phi_1) \in H_1^\rho,\ -h \le \theta \le 0,\ x \in \mathbb{R}^d \end{pmatrix}$$

satisfy the condition

$$\sup_{0\le t\le T} \mathbf{E}\|U(t,\,\cdot\,,\phi) - U(t,\,\cdot\,,\phi_1)\|^2_{H^\rho} \le C(T)\Bigg(\mathbf{E}\|\phi(0,\,\cdot\,) - \phi_1(0,\,\cdot\,)\|^2_{H_0^\rho} + \int_{-h}^{0} \mathbf{E}\|\phi(t,\,\cdot\,) - \phi_1(t,\,\cdot\,)\|^2_{H_0^\rho} dt\Bigg)$$

and the family of operators $P_{s,t}$ *from* (6) *possess the Feller property.*

Next we deal with existence of invariant measures. There is plenty of works, dedicated to this question. Let note the work [5] of Scheutzow et al., where this question is considered in finite dimensional space with finite dimensional Wiener process. The obtained result has been got with the help of Lyapunov functions. Since we deal with infinite dimensional Wiener process and unbounded operator, we have been using an idea of Da Prato et al. [4]. Their approach concerns such equations with no delay. Let us note that general ideas from this approach are valid for our case. But presence of delay makes some technical adjustments to our proof. Therefore, in order to state our result, we have been dealing with the weight function $\rho(x) = \frac{1}{1+|x|^r}$. We present the next theorem, concerning invariant measure existence.

Theorem 4 (invariant measure existence)**.** *If* $r > \overline{r} + d$ *and* (1) *has bounded in probability solution* U *in* $H^{\overline{\rho}}$, *then there exists invariant measure in* H^ρ.

This theorem has one significant moment — checking conditions of existence of bounded solution to our equation on semiaxis $t \ge 0$. The theorem below gives us sufficient coefficient conditions of existence of such a solution.

Theorem 5. *Suppose* $d \ge 3$. *Let* $\sigma\colon H_1^\rho \to H_0^\rho$ *be such that*

$$\|\sigma(u) - \sigma(v)\|_{H_0^\rho} \le L\|u - v\|_{H_1^\rho},\ \{u,v\} \subset H_1^\rho,$$

and for some $\sigma_0 > 0$

$$|\sigma(u)| \le \sigma_0,\ u \in H_1^\rho.$$

Let $f\colon H_1^\rho \to H_0^\rho$ *be such that*

$$\|f(u) - f(v)\|_{H_0^\rho} \le L\|u - v\|_{H_1^\rho},\ \{u, v\} \subset H_1^\rho,$$

and besides for some ψ *from* $L_1(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$ *the following estimate is true*

$$|f(u)| \le \psi(x),\ u \in H_1^\rho,\ x \in \mathbb{R}^d.$$

Then if $u(0,\,\cdot\,) = \phi(0,\,\cdot\,)$ *belongs to* $L_2(\mathbb{R}^d) \cap H_0^\rho$, *then* $\sup\limits_{t\ge 0} \mathbf{E}\|U(t,\,\cdot\,)\|^2_{H^\rho} < \infty$ *.*

3. Preliminaries.

Definition 2. *A nonnegative positive bounded function* ρ *from* $L_1(\mathbb{R}^d)$ *is called **an admissible weight**, if for any* $T > 0$ *there exists* $C_\rho(T) > 0$ *such that for each* $0 \le t \le T$ *the next estimate is true*

$$\int\limits_{\mathbb{R}^d} \mathscr{K}(t, x - \xi)\rho(x)dx \le C_\rho(T)\rho(\xi),\ \ \xi \in \mathbb{R}^d.$$

Remark 2. *As concrete examples of admissible weights we can consider, for instance, the functions* $\rho(x) = \exp\{-r|x|\}$, $x \in \mathbb{R}^d$, $r > 0$, *or* $\rho(x) = \frac{1}{1+|x|^r}$, $x \in \mathbb{R}^d$, *with* $r > d$.

Remark 3. *From now on only admissible weights are considered.*

Remark 4. *For* (C_0)*-semigroup of operators, defined by* (2), *the following estimate is true*

$$\|(S(t)g(\,\cdot\,))(x)\|^2_{H_0^\rho} \le C_\rho(T)\|g(x)\|^2_{H_0^\rho},\ 0 \le t \le T,\ \ g \in H_0^\rho.$$

4.1. Proof of theorem 1 and its corollary 1.

Proof. The proof is based on the classical theorem from functional analysis — Banach theorem on a fixed point. According to it, let's consider $\mathfrak{B}_{p,T,\rho}$ — Banach space of all H_0^ρ-valued $\mathcal{F}_t$-measurable for almost all $0 \le t \le T$ processes $\Phi\colon H_0^\rho \to H_0^\rho$, continuous in t for almost all ω from Ω, with a norm $\|\Phi\|_{\mathfrak{B}_{p,T,\rho}} = \sqrt{\mathbf{E}\int\limits_{-h}^{T} \|\Phi(t,\,\cdot\,)\|^p_{H_0^\rho}dt}$. Let us consider an operator $\Psi\colon \mathfrak{B}_{p,T,\rho} \to \mathfrak{B}_{p,T,\rho}$, acting as follows

$$\Psi u(t,x) = \int\limits_{\mathbb{R}^d} \mathscr{K}(t, x - \xi)\phi(0,\xi)d\xi + \int\limits_0^t\int\limits_{\mathbb{R}^d} \mathscr{K}(t - s, x - \xi)f(u_s(\xi))d\xi ds +$$

$$+ \int\limits_0^t \sum_{n=1}^{\infty} \sqrt{\lambda_n}\Bigg(\int\limits_{\mathbb{R}^d} \mathscr{K}(t - s, x - \xi)\sigma(u_s(\xi))e_n(\xi)d\xi\Bigg)d\beta_n(s),\ 0 \le t \le T,\ x \in \mathbb{R}^d,$$

$$\Psi u(t,x) = \phi(t,x),\ \ -h \le t \le 0,\ x \in \mathbb{R}^d,\ h > 0,$$

$$\Psi u(0,x) = \phi(0,x),\ x \in \mathbb{R}^d.$$

Standard computations yield contraction of Ψ in this space, thereby completing the proof of the theorem.

Next let us show that $\mathbf{E}\|u_t(\,\cdot\,)\|^p_{H_1^\rho} < \infty$, where u_t solves the problem (4) — — (4**). We need to show that

$$\mathbf{E}\|u_t(\,\cdot\,)\|^p_{H_1^\rho} = \mathbf{E}\Bigg(\int\limits_{-h}^{0}\|u(t+\theta,\,\cdot\,)\|^2_{H_0^\rho}d\theta\Bigg)^{\frac{p}{2}} \le h^{\frac{p}{2}-1}\mathbf{E}\int\limits_{-h}^{0}\|u(t+\theta,\,\cdot\,)\|^p_{H_0^\rho}d\theta < \infty.$$

Let us consider separately cases of $0 \le t \le h$ and $h \le t \le T$. For $0 \le t \le h$

$$\mathbf{E}\int\limits_{-h}^{0}\|u(t+\theta,\,\cdot\,)\|^p_{H_0^\rho}d\theta = \mathbf{E}\int\limits_{-h}^{-t}\|u(t+\theta,\,\cdot\,)\|^p_{H_0^\rho}d\theta + \mathbf{E}\int\limits_{-t}^{0}\|u(t+\theta,\,\cdot\,)\|^p_{H_0^\rho}d\theta =$$

$$= \mathbf{E}\int\limits_{t+\theta=t-h}^{t+\theta=0}\|u(t+\theta,\,\cdot\,)\|^p_{H_0^\rho}d(t+\theta) + \mathbf{E}\int\limits_{t+\theta=0}^{t+\theta=t}\|u(t+\theta,\,\cdot\,)\|^p_{H_0^\rho}d(t+\theta) =$$

$$= \mathbf{E}\int\limits_{t-h}^{0}\|u(s,\,\cdot\,)\|^p_{H_0^\rho}ds + \mathbf{E}\int\limits_{0}^{t}\|u(s,\,\cdot\,)\|^p_{H_0^\rho}ds \le$$

$$\le \mathbf{E}\int\limits_{-h}^{0}\|u(s,\,\cdot\,)\|^p_{H_0^\rho}ds + \mathbf{E}\int\limits_{0}^{h}\|u(s,\,\cdot\,)\|^p_{H_0^\rho}ds < \infty.$$

If $h \le t \le T$, then we obtain

$$\mathbf{E}\int\limits_{-h}^{0}\|u(t+\theta,\,\cdot\,)\|^p_{H_0^\rho}d\theta =$$

$$= \mathbf{E}\int\limits_{t+\theta=t-h}^{t+\theta=t}\|u(t+\theta,\,\cdot\,)\|^p_{H_0^\rho}d(t+\theta) \le \mathbf{E}\int\limits_{0}^{T}\|u(s,\,\cdot\,)\|^p_{H_0^\rho}ds < \infty,$$

thereby completing the proof of the corollary. □

4.2. Proof of theorem 2.

Proof. Let $u(t,s,\,\cdot\,,\phi)$ be the solution of (1) — (1*) on $t \ge s$ in the terms of section **2**, i.e., let $u_s(s,\,\cdot\,,\phi) = u(s+\theta, s,\,\cdot\,,\phi) = \phi(s+\theta,\,\cdot\,)$, $-h \le \theta \le 0$, and $u(t,s,\,\cdot\,,\phi)$ satisfy (3) for $t \ge s$. Here the function $\phi(s+\theta,\,\cdot\,,\omega)$ is $\mathcal{F}_s$-measurable and satisfies conditions from assumption 2) for any fixed s such that $0 \le s \le t \le T$. For any fixed s and t $u_t(s,\,\cdot\,,\phi) = u(t+\theta,s,\,\cdot\,,\phi)$ as a function of θ is a random variable from H_1^ρ. Let $\varphi \in H_1^\rho$ be non-random. Then $u(t,s,\,\cdot\,,\varphi)$ is completely defined by the increments $W(\tau) - W(s)$, $\tau \ge s$, therefore it does not depend on σ-algebra $\mathcal{F}_s$ and is $\mathcal{G}_s$-measurable. Here $\mathcal{G}_s$ is the minimal σ-algebra, generated by the increments $W(\tau) - W(s)$, $\tau \ge s$. For any $0 \le s \le \tau \le t \le T$ we have

$$u_t(s,\,\cdot\,,\phi) = u_t(\tau,\,\cdot\,,u_\tau(s,\,\cdot\,,\phi)). \qquad (7)$$

Note that $u_\tau(s,\,\cdot\,,\phi)$ is $\mathcal{F}_\tau$-measurable and does not depend on σ-algebra $\mathcal{G}_\tau$.

Thus, $u_t(s,\,\cdot\,,\phi)=\beta(u_\tau(s,\,\cdot\,,\phi),\omega)$, where $\beta(X,\omega)$, $X\in H_1^\rho$, is a random function that does not depend on events from σ-algebra $\mathcal{F}_\tau$ and is $\mathcal{G}_\tau$-measurable.

In order to prove the theorem, we need to demonstrate that for all $0\le s\le\tau\le\le t\le T$ and $A_1\in\mathfrak{D}_1$ the following identity is true

$$\mathbf{P}\{u_t(s,\,\cdot\,,\phi)\in A_1|\mathcal{F}_\tau\}=P(\tau,u_\tau(s,\,\cdot\,,\phi),t,A_1), \tag{8}$$

where the transition function $P(\tau,u,t,A_1)$ is defined from (5). In order to prove (8) it is enough to show that for any real bounded Borel function $g\colon H_1^\rho\to\mathbb{R}$ we have

$$\mathbf{E}(g(u_t(s,\,\cdot\,,\phi))|\mathcal{F}_\tau)=\mathbf{E}g(u_t(\tau,\,\cdot\,,\varphi))|_{\varphi=u_\tau(s,\,\cdot\,,\phi)}. \tag{9}$$

(9) is proved similarly to [4, theorem 3.8] with the help of (7), independence of $u_t(\tau,\,\cdot\,,\varphi)$ from σ-algebra $\mathcal{F}_\tau$ and its $\mathcal{F}_\tau$-measurability. The theorem is proved. $\square$

4.3. Proof of theorem 3.

Proof. Let ϕ and ϕ_1 satisfy conditions from 2) and let $u(t,x,\phi)$ and $u(t,x,\phi_1)$ be solutions, corresponding to initial datums ϕ and ϕ_1, respectively. We have

$$\sup_{0\le t\le T}\mathbf{E}\|U(t,\,\cdot\,,\phi)-U(t,\,\cdot\,,\phi_1)\|^2_{H^\rho}=\sup_{0\le t\le T}\mathbf{E}\Bigg(\|u(t,\,\cdot\,,\phi)-u(t,\,\cdot\,,\phi_1)\|^2_{H_0^\rho}+$$

$$+\sup_{0\le t\le T}\mathbf{E}\|u_t(\,\cdot\,,\phi)-u_t(\,\cdot\,,\phi_1)\|^2_{H_1^\rho}\Bigg)\le\sup_{0\le t\le T}\mathbf{E}\|u(t,\,\cdot\,,\phi)-u(t,\,\cdot\,,\phi_1)\|^2_{H_0^\rho}+$$

$$+\sup_{0\le t\le T}\mathbf{E}\int_{-h}^{0}\|u(t+\theta,\,\cdot\,,\phi)-u(t+\theta,\,\cdot\,,\phi_1)\|^2_{H_0^\rho}d\theta. \tag{10}$$

Taking into account the identity

$$u(t,x,\phi)-u(t,x,\phi_1)=\int_{\mathbb{R}^d}\mathcal{K}(t,x-\xi)\big(\phi(0,\xi)-\phi_1(0,\xi)\big)d\xi+$$

$$+\int_0^t\int_{\mathbb{R}^d}\mathcal{K}(t-s,x-\xi)\big(f(u(s+\theta,\xi,\phi))-f(u(s+\theta,\xi,\phi_1))\big)d\xi ds+$$

$$+\int_0^t\sum_{n=1}^{\infty}\sqrt{\lambda_n}\Bigg(\int_{\mathbb{R}^d}\mathcal{K}(t-s,x-\xi)\big(\sigma(u(s+\theta,\xi,\phi))-\sigma(u(s+\theta,\xi,\phi_1))\big)e_n(\xi)d\xi\Bigg)d\beta_n(s)$$

$$0\le t\le T,\ -h\le\theta\le0,\ x\in\mathbb{R}^d,$$

we obtain by standard computations, using Gronwall's inequality, the following estimate for $\sup\limits_{0\le t\le T}\mathbf{E}\|u(t,\,\cdot\,,\phi)-u(t,\,\cdot\,,\phi_1)\|^2_{H_0^\rho}$

$$\sup_{0\le t\le T}\mathbf{E}\|u(t,\,\cdot\,,\phi)-u(t,\,\cdot\,,\phi_1)\|^2_{H_0^\rho}\le\Bigg(3C_\rho(T)\mathbf{E}\|\phi(0,\,\cdot\,)-\phi_1(0,\,\cdot\,)\|^2_{H_0^\rho}+$$

$$+3L^2C_\rho(T)h\Big(T+\sum_{n=1}^{\infty}\lambda_n\Big)\mathbf{E}\int\limits_{-h}^{0}\|\phi(s,\,\cdot\,)-\phi_1(s,\,\cdot\,)\|^2_{H_0^\rho}ds\Big)\times$$

$$\times\exp\left\{3L^2C_\rho(T)\Big(T+\sum_{n=1}^{\infty}\lambda_n\Big)T\cdot T\right\}\le$$

$$\le C(T)\Big(\mathbf{E}\|\phi(0,\,\cdot\,)-\phi_1(0,\,\cdot\,)\|^2_{H_0^\rho}+\mathbf{E}\int\limits_{-h}^{0}\|\phi(t,\,\cdot\,)-\phi_1(t,\,\cdot\,)\|^2_{H_0^\rho}dt\Big), \tag{11}$$

$$C(T)=3C_\rho(T)\max\left\{1;hL^2\Big(T+\sum_{n=1}^{\infty}\lambda_n\Big)\exp\left\{3L^2C_\rho(T)\Big(T+\sum_{n=1}^{\infty}\lambda_n\Big)T\cdot T\right\}\right\}.$$

Next let us estimate $\sup\limits_{0\le t\le T}\mathbf{E}\int\limits_{-h}^{0}\|u(t+\theta,\,\cdot\,,\phi)-u(t+\theta,\,\cdot\,,\phi_1)\|^2_{H_0^\rho}d\theta$. We get

$$\sup_{0\le t\le T}\mathbf{E}\int\limits_{-h}^{0}\|u(t+\theta,\,\cdot\,,\phi)-u(t+\theta,\,\cdot\,,\phi_1)\|^2_{H_0^\rho}d\theta$$

$$\le\sup_{0\le t\le h}\mathbf{E}\int\limits_{-h}^{0}\|u(t+\theta,\,\cdot\,,\phi)-u(t+\theta,\,\cdot\,,\phi_1)\|^2_{H_0^\rho}d\theta$$

$$+\sup_{h\le t\le T}\mathbf{E}\int\limits_{-h}^{0}\|u(t+\theta,\,\cdot\,,\phi)-u(t+\theta,\,\cdot\,,\phi_1)\|^2_{H_0^\rho}d\theta. \tag{12}$$

Taking into account estimate for $\mathbf{E}\|u(t,\,\cdot\,,\phi)-u(t,\,\cdot\,,\phi_1)\|^2_{H_0^\rho}$, $0\le t\le T$, following from (11), we have

$$\sup_{0\le t\le h}\mathbf{E}\int\limits_{-h}^{0}\|u(t+\theta,\,\cdot\,,\phi)-u(t+\theta,\,\cdot\,,\phi_1)\|^2_{H_0^\rho}d\theta=$$

$$=\sup_{0\le t\le h}\mathbf{E}\int\limits_{-h}^{-t}\|u(t+\theta,\,\cdot\,,\phi)-u(t+\theta,\,\cdot\,,\phi_1)\|^2_{H_0^\rho}d\theta+$$

$$+\sup_{0\le t\le h}\mathbf{E}\int\limits_{-t}^{0}\|u(t+\theta,\,\cdot\,,\phi)-u(t+\theta,\,\cdot\,,\phi_1)\|^2_{H_0^\rho}d\theta\le$$

$$\le\mathbf{E}\int\limits_{-h}^{0}\|\phi(s,\,\cdot\,)-\phi_1(s,\,\cdot\,)\|^2_{H_0^\rho}ds+\mathbf{E}\int\limits_{0}^{T}\|u(s,\,\cdot\,,\phi)-u(s,\,\cdot\,,\phi_1)\|^2_{H_0^\rho}ds\le$$

$$\le\mathbf{E}\int\limits_{-h}^{0}\|\phi(s,\,\cdot\,)-\phi_1(s,\,\cdot\,)\|^2_{H_0^\rho}ds+C(T)T\Big(\mathbf{E}\|\phi(0,\,\cdot\,)-\phi_1(0,\,\cdot\,)\|^2_{H_0^\rho}+$$

$$+\mathbf{E}\int\limits_{-h}^{0}\|\phi(s,\,\cdot\,)-\phi_1(s,\,\cdot\,)\|^2_{H_0^\rho}ds\Bigg), \tag{13}$$

$$\sup_{h\le t\le T}\mathbf{E}\int\limits_{-h}^{0}\|u(t+\theta,\,\cdot\,,\phi)-u(t+\theta,\,\cdot\,,\phi_1)\|^2_{H_0^\rho}d\theta\le\mathbf{E}\int\limits_{0}^{T}\|u(s,\,\cdot\,,\phi)-u(s,\,\cdot\,,\phi_1)\|^2_{H_0^\rho}ds\le$$

$$\le C(T)T\Bigg(\mathbf{E}\|\phi(0,\,\cdot\,)-\phi_1(0,\,\cdot\,)\|^2_{H_0^\rho}+\mathbf{E}\int\limits_{-h}^{0}\|\phi(s,\,\cdot\,)-\phi_1(s,\,\cdot\,)\|^2_{H_0^\rho}ds\Bigg). \tag{14}$$

Thus, we obtain from (12), applying (13) and (14),

$$\sup_{0\le t\le T}\mathbf{E}\int\limits_{-h}^{0}\|u(t+\theta,\,\cdot\,,\phi)-u(t+\theta,\,\cdot\,,\phi_1)\|^2_{H_0^\rho}d\theta\le\mathbf{E}\int\limits_{-h}^{0}\|\phi(s,\,\cdot\,)-\phi_1(s,\,\cdot\,)\|^2_{H_0^\rho}ds$$

$$+\,2C(T)T\Bigg(\mathbf{E}\|\phi(0,\,\cdot\,)-\phi_1(0,\,\cdot\,)\|^2_{H_0^\rho}+\mathbf{E}\int\limits_{-h}^{0}\|\phi(s,\,\cdot\,)-\phi_1(s,\,\cdot\,)\|^2_{H_0^\rho}ds\Bigg), \tag{15}$$

and from (16) with the help of (11) and (15) we derive

$$\sup_{0\le t\le T}\mathbf{E}\|U(t,\,\cdot\,,\phi)-U(t,\,\cdot\,,\phi_1)\|^2_H\le C(T)\Bigg(\mathbf{E}\|\phi(0,\,\cdot\,)-\phi_1(0,\,\cdot\,)\|^2_{H_0^\rho}+$$

$$+\mathbf{E}\int\limits_{-h}^{0}\|\phi(s,\,\cdot\,)-\phi_1(s,\,\cdot\,)\|^2_{H_0^\rho}ds\Bigg)+\mathbf{E}\int\limits_{-h}^{0}\|\phi(s,\,\cdot\,)-\phi_1(s,\,\cdot\,)\|^2_{H_0^\rho}ds+$$

$$+2C(T)T\Bigg(\mathbf{E}\|\phi(0,\,\cdot\,)-\phi_1(0,\,\cdot\,)\|^2_{H_0^\rho}+\mathbf{E}\int\limits_{-h}^{0}\|\phi(s,\,\cdot\,)-\phi_1(s,\,\cdot\,)\|^2_{H_0^\rho}ds\Bigg)\le$$

$$\le\big(C(T)(2T+1)+1\big)\Bigg(\mathbf{E}\|\phi(0,\,\cdot\,)-\phi_1(0,\,\cdot\,)\|^2_{H_0^\rho}+\mathbf{E}\int\limits_{-h}^{0}\|\phi(s,\,\cdot\,)-\phi_1(s,\,\cdot\,)\|^2_{H_0^\rho}ds\Bigg),$$

thereby completing the proof of the theorem. □

4.4. Proof of theorem 4 (sketch)**.** Based on the classic scheme [4, theorem 11.29], adapted to functional-differential equations, and the standard technique from Krylov–Bogolyubov theorem, we now prove our theorem for the existence of an invariant measure. This adaptation is based on three lemmas. Let $T_0>2r$ be fixed.

Lemma 1. *An operator* $A\colon H_0^{\overline{\rho}}\to H_1^\rho$, *acting as* $A\varphi_0(\,\cdot\,)=S(T_0+\theta)\varphi_0(\,\cdot\,)$, *is a Hilbert-Schmidt operator.*

Proof. Similarly to [3], if $\{e_n(x), n \in \{1, 2, \dots\}\}$ is an orthonormal basis on $L_2(\mathbb{R}^d)$, then $\left\{\overline{\rho}^{-\frac{1}{2}}(x)e_n(x), n \in \{1, 2, \dots\}\right\}$ is an orthonormal basis on $H_0^{\overline{\rho}}$. Indeed, we obtain for Hilbert-Schmidt norm $\|\cdot\|_{HS}$ of operator A

$$\|A\|_{HS}^2 = \sum_{j=1}^{\infty} \left\| A \frac{e_j(\cdot)}{\sqrt{\overline{\rho}(\cdot)}} \right\|_{H_1^{\rho}}^2 = \int_{-h}^{0} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{\mathscr{K}^2(T_0+\theta, x-\xi)}{\sqrt{\overline{\rho}(\xi)}} d\xi \right) \rho(x) dx \right) d\theta \leq$$

$$\leq \int_{-h}^{0} \frac{1}{\left(8\pi(T_0+\theta)\right)^{\frac{d}{2}}} C_\rho \left(\frac{T_0+\theta}{2} \right) \left(\int_{\mathbb{R}^d} \frac{1+|x|^{\overline{\rho}}}{1+|x|^{\rho}} dx \right) d\theta < \infty.$$

□

Next we have

$$u(T_0, \cdot) = S(T_0)\varphi(\cdot) + \int_0^{T_0} S(T_0 - s) f(u_s(\cdot)) ds + \int_0^{T_0} S(T_0 - s)\sigma(u_s(\cdot)) dW(s, \cdot),$$

$$u_{T_0}(\cdot) = u(T_0+\theta, \cdot) = S(T_0+\theta)\varphi(\cdot) + \int_0^{T_0+\theta} S(T_0+\theta-s) f(u_s(\cdot)) ds +$$

$$+ \int_0^{T_0+\theta} S(T_0+\theta-s)\sigma(u_s(\cdot)) dW(s, \cdot).$$

Let us take $\frac{1}{p} < \alpha < \frac{1}{2}$, $p > 2$, and let φ belong to the space $L_p([0, T_0]; H_0^{\overline{\rho}})$. We consider an operator $(G_\alpha \varphi(\cdot))(\theta) = \int\limits_0^{T_0+\theta} (T_0+\theta-s)^{\alpha-1} S(T_0+\theta-s)\varphi(s) ds$ from $C([-h; 0]; H_0^{\rho})$. Using an infinite dimensional version of Arzelȧ-Askoli theorem, we have proved the following result.

Lemma 2. *G_α is a compact operator, acting from $L_p([0, T_0]; H_0^{\overline{\rho}})$ into $C([-h; 0]; H_0^{\rho})$.*

Next we consider a set $K(r) \subset H^\rho$, $K(r) = \{(x, z) : x \in H_0^\rho,\ z \in H_1^\rho,\ x = S(T_0)\nu(\cdot) + (G_1\varphi(\cdot))(0) + (G_\alpha h(\cdot))(0),\ z = S(T_0+\theta)\nu(\cdot) + (G_1\varphi(\cdot))(\theta) + (G_\alpha h(\cdot))(\theta),\ \|\nu(\cdot)\|_{H_0^{\overline{\rho}}} \leq r,\ \|\varphi(\cdot)\|_{L_p([0,T_0]; H_0^{\overline{\rho}})} \leq r,\ \|h(\cdot)\|_{L_p([0,T_0]; H_0^{\overline{\rho}})} \leq r\}$. In virtue of lemma 2, $K(r)$ is compact in H^ρ.

Lemma 3. *Let conditions from* 1) be true. Then there exists a constant $C > 0$ such that for any $r > 0$ and $y = \begin{pmatrix} x \\ z \end{pmatrix} \in H^\rho$ with $\|y(\cdot)\|_{H^\rho} \leq r$ we have $\mathbf{P}\{(u(T_0, x, z), u_{T_0}(x, z)) \in K(r)\} \geq 1 - Cr^{-p}\left(1 + \|y(\cdot)\|_{H^\rho}^p\right)$.

Proof. We skip the proof and just mention that the result has been obtained with the help of standard estimates along with the factorisation method. □

Similarly to [4, theorem 11.29] with the help of lemmas 1 — 3 it is possible to prove that there exists $r > 0$ such that for any $\epsilon > 0$ $\mathbf{P}\{(u(t, \cdot), u_t(\cdot)) \in K(r)\} > 1 - r$. Thus, we have obtained the tightness of a family of measures $\mathcal{L}(u(t, \cdot), u_t(\cdot))$

for $t > 1$. Based on this result, Krylov-Bogoliuibov procedure yields our theorem. □

4.5. Proof of theorem 5.

Our aim is to show that

$$\sup_{t\geq 0}\mathbf{E}\|U(t,\,\cdot\,)\|^2_{H^\rho}=\sup_{t\geq 0}\mathbf{E}\Bigg(\|u(t,\,\cdot\,)\|^2_{H^\rho_0}+\|u_t(\,\cdot\,)\|^2_{H^\rho_1}\Bigg)$$

$$\leq\sup_{t\geq 0}\mathbf{E}\|u(t,\,\cdot\,)\|^2_{H^\rho_0}+\sup_{t\geq 0}\mathbf{E}\int\limits_{-h}^{0}\|u(t+\theta,\,\cdot\,)\|^2_{H^\rho_0}d\theta<\infty.$$

Firstly let us show $\sup\limits_{t\geq 0}\mathbf{E}\|u(t,\,\cdot\,)\|^2_{H^\rho_0}<\infty$. We begin by observing that

$$\sup_{t\geq 0}\mathbf{E}\|u(t,\,\cdot\,)\|^2_{H^\rho_0}=\sup_{t\geq 0}\mathbf{E}\Bigg\|S(t)\phi(0,\,\cdot\,)+\int\limits_0^t S(t-s)f(u(s+\theta,\,\cdot\,))ds+$$

$$+\int\limits_0^t S(t-s)\sigma(u(s+\theta,\,\cdot\,))dW(s,\,\cdot\,)\Bigg\|^2_{H^\rho_0}=\sup_{t\geq 0}\mathbf{E}\Bigg\|\sum_{j=0}^{2}J_j(t)\Bigg\|^2_{H^\rho_0}\leq$$

$$\leq 3\sum_{j=0}^{2}\sup_{t\geq 0}\mathbf{E}\|J_j(t)\|^2_{H^\rho_0}. \tag{16}$$

Let us estimate $\sup\limits_{t\geq 0}\mathbf{E}\|J_j(t)\|^2_{H^\rho_0}$, $j\in\{0;1;2\}$, separately.

$$\sup_{t\geq 0}\mathbf{E}\|J_0(t)\|^2_{H^\rho_0}=\sup_{t\geq 0}\mathbf{E}\|S(t)\phi(0,\,\cdot\,)\|^2_{H^\rho_0}=$$

$$=\sup_{t\geq 0}\mathbf{E}\int\limits_{\mathbb{R}^d}\Bigg(\int\limits_{\mathbb{R}^d}\mathscr{K}(t,x-\xi)\phi(0,\xi)d\xi\Bigg)^2\rho(x)dx\leq$$

$$\leq\sup_{t\geq 0}\mathbf{E}\int\limits_{\mathbb{R}^d}\Bigg(\int\limits_{\mathbb{R}^d}\mathscr{K}(t,x-\xi)d\xi\Bigg)\Bigg(\int\limits_{\mathbb{R}^d}\mathscr{K}(t,x-\xi)\phi^2(0,\xi)d\xi\Bigg)\rho(x)dx=$$

$$=\sup_{t\geq 0}\mathbf{E}\int\limits_{\mathbb{R}^d}\Bigg(\int\limits_{\mathbb{R}^d}\mathscr{K}(t,x-\xi)\rho(x)dx\Bigg)\phi^2(0,\xi)d\xi\leq$$

$$\leq\Bigg(ess\sup_{x\in\mathbb{R}^d}\rho(x)\Bigg)\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}<\infty.$$

$$\sup_{t\geq 0}\mathbf{E}\|J_1(t)\|^2_{H^\rho_0}=\sup_{t\geq 0}\mathbf{E}\Bigg\|\int\limits_0^t S(t-s)f(u(s+\theta,\,\cdot\,))ds\Bigg\|^2_{H^\rho_0}=$$

$$=\sup_{t\geq 0}\mathbf{E}\int\limits_{\mathbb{R}^d}\Bigg(\int\limits_0^t\int\limits_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)f(u(s+\theta,\xi))d\xi ds\Bigg)^2\rho(x)dx\leq$$

$$\leq 2\sup_{t\geq 0}\Bigg(\mathbf{E}\int\limits_{\mathbb{R}^d}\Big(\int\limits_0^{t-1}\int\limits_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)f(u(s+\theta,\xi))d\xi ds\Big)^2\rho(x)dx+$$

$$+\mathbf{E}\int\limits_{\mathbb{R}^d}\Big(\int\limits_{t-1}^{t}\int\limits_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)f(u(s+\theta,\xi))d\xi ds\Big)^2\rho(x)dx\Bigg)=$$

$$=2\sup_{t\geq 0}\big(J_1^1(t)+J_1^2(t)\big). \tag{17}$$

$$J_1^2(t)=\mathbf{E}\int\limits_{\mathbb{R}^d}\Big(\int\limits_{t-1}^{t}\int\limits_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)f(u(s+\theta,\xi))d\xi ds\Big)^2\rho(x)dx\leq$$

$$\leq\mathbf{E}\int\limits_{\mathbb{R}^d}\Big(\int\limits_{t-1}^{t}\Big(\int\limits_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)f(u(s+\theta,\xi))d\xi\Big)^2ds\Big)\rho(x)dx\leq$$

$$\leq\mathbf{E}\int\limits_{\mathbb{R}^d}\Big(\int\limits_{t-1}^{t}\Big(\int\limits_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)f^2(u(s+\theta,\xi))d\xi\Big)ds\Big)\rho(x)dx\leq$$

$$\leq\Big(ess\sup_{x\in\mathbb{R}^d}\psi^2(x)\Big)\Big(\int\limits_{\mathbb{R}^d}\rho(x)dx\Big)<\infty, \tag{18}$$

$$J_1^1(t)\leq\mathbf{E}\int\limits_{\mathbb{R}^d}\Big(\int\limits_0^{t-1}\int\limits_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)|f(u(s+\theta,\xi))|d\xi ds\Big)^2\rho(x)dx=$$

$$=\mathbf{E}\int\limits_{\mathbb{R}^d}\Big(\int\limits_0^{t-1}\frac{1}{\big(4\pi(t-s)\big)^{\frac{d}{2}}}\Big(\int\limits_{\mathbb{R}^d}\exp\Big\{-\frac{|x-\xi|^2}{4(t-s)}\Big\}|f(u(s+\theta,\xi))|d\xi\Big)ds\Big)^2\rho(x)dx\leq$$

$$\leq\mathbf{E}\int\limits_{\mathbb{R}^d}\Big(\int\limits_0^{t-1}\frac{1}{\big(4\pi(t-s)\big)^{\frac{d}{2}}}\Big(\int\limits_{\mathbb{R}^d}\psi(\xi)d\xi\Big)ds\Big)^2\rho(x)dx=$$

$$=\frac{1}{(4\pi)^d}\Big(\int\limits_{\mathbb{R}^d}\psi(\xi)d\xi\Big)^2\Big(\int\limits_{\mathbb{R}^d}\rho(x)dx\Big)\Big(\int\limits_0^{t-1}\frac{1}{(t-s)^{\frac{d}{2}}}ds\Big)^2=$$

$$=\frac{1}{(4\pi)^d}\Big(\int\limits_{\mathbb{R}^d}\psi(\xi)d\xi\Big)^2\Big(\int\limits_{\mathbb{R}^d}\rho(x)dx\Big)\Big(\int\limits_1^{t}\frac{1}{\tau^{\frac{d}{2}}}d\tau\Big)^2. \tag{19}$$

Thus, we have from (17), using (19) and (18), the estimate $\sup\limits_{t\geq 0}\mathbf{E}\|J_1(t)\|^2_{H_0^\rho}<\infty$.

$$\sup_{t\geq 0}\mathbf{E}\|J_2(t)\|^2_{H_0^\rho}=\sup_{t\geq 0}\mathbf{E}\Big\|\int\limits_0^t S(t-s)\sigma(u(s+\theta,\cdot))dW(s,\cdot)\Big\|^2_{H_0^\rho}=$$

$$= \sup_{t \geq 0} \mathbf{E} \int_{\mathbb{R}^d} \Big(\int_0^t \int_{\mathbb{R}^d} \mathscr{K}(t-s, x-\xi) \sigma(u(s+\theta, \xi)) d\xi dW(s,x) \Big)^2 \rho(x) dx. \tag{20}$$

Let us consider the integrand from (20) separately.

$$\mathbf{E} \Big(\int_0^t \int_{\mathbb{R}^d} \mathscr{K}(t-s, x-\xi) \sigma(u(s+\theta, \xi)) d\xi dW(s,x) \Big)^2 =$$

$$= \mathbf{E} \Big(\int_0^t \sum_{n=1}^{\infty} \sqrt{\lambda_n} \Big(\int_{\mathbb{R}^d} \mathscr{K}(t-s, x-\xi) \sigma(u(s+\theta, \xi)) e_n(\xi) d\xi \Big) d\beta_n(s) \Big)^2 =$$

$$= \mathbf{E} \int_0^t \sum_{n=1}^{\infty} \lambda_n \Big(\int_{\mathbb{R}^d} \mathscr{K}(t-s, x-\xi) \sigma(u(s+\theta, \xi)) e_n(\xi) d\xi \Big)^2 ds =$$

$$= \mathbf{E} \int_0^{t-1} \sum_{n=1}^{\infty} \lambda_n \Big(\int_{\mathbb{R}^d} \mathscr{K}(t-s, x-\xi) \sigma(u(s+\theta, \xi)) e_n(\xi) d\xi \Big)^2 ds +$$

$$+ \mathbf{E} \int_{t-1}^{t} \sum_{n=1}^{\infty} \lambda_n \Big(\int_{\mathbb{R}^d} \mathscr{K}(t-s, x-\xi) \sigma(u(s+\theta, \xi)) e_n(\xi) d\xi \Big)^2 ds = J_2^1(t) + J_2^2(t). \tag{21}$$

$$J_2^1(t) \leq \mathbf{E} \int_0^{t-1} \sum_{n=1}^{\infty} \lambda_n \Big(\int_{\mathbb{R}^d} \mathscr{K}(t-s, x-\xi) |\sigma(u(s+\theta, \xi))| e_n(\xi) d\xi \Big)^2 ds \leq$$

$$\leq \sigma_0^2 \int_0^{t-1} \sum_{n=1}^{\infty} \lambda_n \Big(\int_{\mathbb{R}^d} \mathscr{K}(t-s, x-\xi) |e_n(\xi)| d\xi \Big)^2 ds \leq$$

$$\leq \sigma_0^2 \int_0^{t-1} \sum_{n=1}^{\infty} \lambda_n \Big(\int_{\mathbb{R}^d} \mathscr{K}^2(t-s, x-\xi) d\xi \Big) ds =$$

$$= \sigma_0^2 \int_1^{t} \sum_{n=1}^{\infty} \lambda_n \Big(\int_{\mathbb{R}^d} \mathscr{K}^2(\tau, x-\xi) d\xi \Big) d\tau. \tag{22}$$

$$J_2^2(t) \leq \mathbf{E} \int_{t-1}^{t} \sum_{n=1}^{\infty} \lambda_n \Big(\int_{\mathbb{R}^d} \mathscr{K}(t-s, x-\xi) |\sigma(u(s+\theta, \xi))| e_n(\xi) d\xi \Big)^2 ds \leq$$

$$\leq \sigma_0^2 \int_{t-1}^{t} \sum_{n=1}^{\infty} \lambda_n \Big(\int_{\mathbb{R}^d} \mathscr{K}(t-s, x-\xi) d\xi \Big)^2 ds = \sigma_0^2 \sum_{n=1}^{\infty} \lambda_n. \tag{23}$$

Then estimates (20), (21) with (22) and (23) together imply

$$\sup_{t\geq 0}\mathbf{E}\|J_2(t)\|^2_{H_0^\rho}\leq \sup_{t\geq 0}\mathbf{E}\int\limits_{\mathbb{R}^d}\Bigg(\sigma_0^2\int\limits_1^t\sum_{n=1}^{\infty}\lambda_n\Big(\int\limits_{\mathbb{R}^d}\mathscr{K}^2(\tau,x-\xi)d\xi\Big)d\tau+$$

$$+\sigma_0^2\sum_{n=1}^{\infty}\lambda_n\Bigg)\rho(x)dx=\sigma_0^2\Big(\sum_{n=1}^{\infty}\lambda_n\Big)\Big(\int\limits_{\mathbb{R}^d}\rho(x)dx\Big)\Big(1+\int\limits_1^{\infty}\Big(\int\limits_{\mathbb{R}^d}\mathscr{K}^2(\tau,y)dy\Big)d\tau\Big).$$

We next prove $\int\limits_1^{\infty}\Big(\int\limits_{\mathbb{R}^d}\mathscr{K}^2(\tau,y)dy\Big)d\tau<\infty$. We have

$$\int\limits_1^{\infty}\Big(\int\limits_{\mathbb{R}^d}\mathscr{K}^2(\tau,y)dy\Big)d\tau=\int\limits_1^{\infty}\frac{1}{(4\pi\tau)^{\frac{d}{2}}}\Big(\int\limits_{\mathbb{R}^d}\frac{1}{(4\pi\tau)^{\frac{d}{2}}}\exp\Big\{-\frac{|y|^2}{4\tau}\Big\}\exp\Big\{-\frac{|y|^2}{4\tau}\Big\}dy\Big)d\tau\leq$$

$$\leq\int\limits_1^{\infty}\frac{1}{(4\pi\tau)^{\frac{d}{2}}}\Big(\int\limits_{\mathbb{R}^d}\frac{1}{(4\pi\tau)^{\frac{d}{2}}}\exp\Big\{-\frac{|y|^2}{4\tau}\Big\}dy\Big)d\tau=\int\limits_1^{\infty}\frac{1}{(4\pi\tau)^{\frac{d}{2}}}\Big(\int\limits_{\mathbb{R}^d}\mathscr{K}(\tau,y)dy\Big)d\tau=$$

$$=\frac{1}{2^d\pi^{\frac{d}{2}}}\int\limits_1^{\infty}\frac{1}{\tau^{\frac{d}{2}}}d\tau<\infty.$$

Using the obtained estimates along with (16) gives rise to $\sup\limits_{t\geq 0}\mathbf{E}\|u(t,\cdot)\|^2_{H_0^\rho}<\infty$.

It remains to show $\sup\limits_{t\geq 0}\mathbf{E}\int\limits_{-h}^{0}\|u(t+\theta,\cdot)\|^2_{H_0^\rho}d\theta<\infty$.

We get

$$\sup_{t\geq 0}\mathbf{E}\int\limits_{-h}^{0}\|u(t+\theta,\cdot)\|^2_{H_0^\rho}d\theta\leq$$

$$\leq\sup_{0\leq t\leq h}\mathbf{E}\int\limits_{-h}^{0}\|u(t+\theta,\cdot)\|^2_{H_0^\rho}d\theta+\sup_{t\geq h}\mathbf{E}\int\limits_{-h}^{0}\|u(t+\theta,\cdot)\|^2_{H_0^\rho}d\theta.$$

It follows from the obtained above estimate $\mathbf{E}\|u(t,\cdot)\|^2_{H_0^\rho}<\infty$, $0\leq t\leq h$. In virtue of this result for the first term we conclude

$$\sup_{0\leq t\leq h}\mathbf{E}\int\limits_{-h}^{0}\|u(t+\theta,\cdot)\|^2_{H_0^\rho}d\theta=\sup_{0\leq t\leq h}\mathbf{E}\int\limits_{-h}^{-t}\|u(t+\theta,\cdot)\|^2_{H_0^\rho}d\theta+$$

$$+\sup_{0\leq t\leq h}\mathbf{E}\int\limits_{-t}^{0}\|u(t+\theta,\cdot)\|^2_{H_0^\rho}d\theta\leq\mathbf{E}\int\limits_{-h}^{0}\|\phi(s,\cdot)\|^2_{H_0^\rho}ds+\mathbf{E}\int\limits_{0}^{h}\|u(s,\cdot)\|^2_{H_0^\rho}ds<\infty.$$

For the second term we obtain

$$\sup_{t\geq h}\mathbf{E}\int\limits_{-h}^{0}\|u(t+\theta,\cdot)\|^2_{H_0^\rho}d\theta\leq\sup_{t\geq h}\mathbf{E}\int\limits_{0}^{t}\|u(s,\cdot)\|^2_{H_0^\rho}ds\leq\mathbf{E}\int\limits_{0}^{h}\|u(s,\cdot)\|^2_{H_0^\rho}ds+$$

$$+\sup_{t\geq h}\mathbf{E}\int_h^t \|u(s,\,\cdot\,)\|^2_{H_0^\rho}ds=\mathbf{E}\int_0^h \|u(s,\,\cdot\,)\|^2_{H_0^\rho}ds+\sup_{t\geq h}\mathbf{E}\int_h^t \left\|\sum_{j=0}^{2} J_j(s)\right\|^2_{H_0^\rho}ds\leq$$

$$\leq \mathbf{E}\int_0^h \|u(s,\,\cdot\,)\|^2_{H_0^\rho}ds+3\sum_{j=0}^{2}\sup_{t\geq h}\mathbf{E}\int_h^t \|J_j(s)\|^2_{H_0^\rho}ds.$$

Arguing as above, we estimate $\sup\limits_{t\geq h}\mathbf{E}\int\limits_h^t \|J_j(s)\|^2_{H_0^\rho}ds$, $j\in\{0;1;2\}$, similarly to $\sup\limits_{t\geq 0}\mathbf{E}\|J_j(t)\|^2_{H_0^\rho}$, $j\in\{0;1;2\}$, thereby completing the proof. □

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Одержано 27.09.2018