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P. I. Stetsyuk[1], O. S. Pichugina[2]

[1] V. M. Glushkov Institute of Cybernetics,
Chair of Department Of Nonsmooth Optimization,
Doctor of Science in Physics and Mathematics
stetsyukp@gmail.com
ORCID: https://orcid.org/0000-0003-4036-2543

[2] National Aerospace University "Kharkiv Aviation Institute",
Associate Professor at Department of Mathematical Modeling and Artificial Intelligence,
Candidate of Science in Physics and Mathematics
oksanapichugina1@gmail.com
ORCID: https://orcid.org/0000-0002-7099-8967

SHOR'S BOUNDS FOR THE WEIGHTED INDEPENDENCE NUMBER

Application of a technique of dual Lagrangian quadratic bounds of N.Z. Shor to studying the Maximum Weighted Independent Set problem is described. By the technique, two such N.Z. Shor's upper bounds are obtained. These are bounds of the graph weighted independence number $\alpha(G, w)$, which can be found in polynomial time. The first bound $\psi(G, w)$ is associated with a quadratic model of the Maximum Weighted Independent Set problem and coincides with the known Lovász number $\vartheta(G, w)$. The second bound $\psi_1(G, w)$ corresponds to the same quadratic model supplemented by a family of functionally redundant quadratic constraints and is able to improve the accuracy of the upper bound $\alpha(G, w)$ for special graph families. It is shown that, if graph is bipartite or perfect, $\psi(G, w) = \alpha(G, w)$, while $\psi_1(G, w) = \alpha(G, w)$ for t- or W_p-perfect graphs. Based on the graph classes that were singled out, a technique is demonstrated, which enables us to form new classes of graphs for which polynomial solvability of the Maximum Weighted Independent Set problem is preserved. Thus, by an example of the Maximum Weighted Independent Set problem in a graph, it is shown how the Lagrangian bounds' technique can be applied to solving an issue of single outing new classes of polynomial solvable combinatorial optimization problems. This approach can be used for improving known bounds of the objective function in combinatorial optimization problems as well as for justifying their polynomial solvability.

Keywords: Maximum Weighted Independent Set problem, the graph weighted independence number, quadratic optimization, Lagrangian dual bounds, polyhedral relaxation, perfect graph, bipartite graph.

1. Introduction. A great contribution to the development of Computational Complexity Theory for continuous and discrete optimization problems was made by N. Z. Shor, who proposed a technique of Lagrangian (dual) bounds on the global extremum in non-convex quadratic problems [1]. In minimization problems, these will be lower bounds while in maximization problems, these will be upper bounds. This technique includes algorithms for finding Lagrangian bounds based on applying non-differentiable optimization and utilizing functionally redundant constraints to improve the Lagrangian bounds' accuracy. The Lagrangian bounds technique can be used to find effective bounds of the global extremum of objective functions in multi-objective optimization problems, which can be formulated as non-convex quadratic problems. We can single out subclasses of NP-hard problems solvable in polynomial time through application of the technique.

The technique of Lagrangian bounds plays a significant role in a study of NP-hard combinatorial Boolean problems. A Boolean constraint on the variable $x \in \{0,1\}$ is representable by a quadratic equality $x^2 - x = 0$. Such formulations of combinatorial problems as nonlinear quadratic problems allow obtaining more accurate dual estimates than the ones that can be derived by relaxing the Boolean optimization problems to the corresponding linear programs. In a monograph [1], this is illustrated by extremal graph problems such as the Maximum Weight Independent Set problem, the Maximum Cut problem, the Maximum Graph Bisection problem, the Minimal k-partition problem, etc. Very interesting were the results for the problem of finding the Maximum Independent (Stable) Set problem, where the dual bounds obtained by N. Z. Shor are closely related to the well-known Lovasz numbers $\vartheta(G,w)$ and $\vartheta'(G,w)$ [2].

In the paper, we continue a study [3] and present new properties of Shor's bounds for the weighted independence number of undirected graphs. It will be shown that the technique of Lagrangian bounds allows single outing such classes of this NP-hard problem that are solvable in polynomial time. The order of presenting the material will be as follows. First, we describe a method for evaluating upper Lagrangian bounds of objective functions in general quadratic maximization problems. Next, we demonstrate that functionally redundant constraints are able to improve the accuracy of these bounds. Then, the Maximum Independent Set problem will be considered in detail, and the difficulty of its solving will be highlighted. It will be followed by presenting an improved Shor's bound for bipartite and perfect graphs and establishing its connection with the Lovasz numbers. Then, an improved Shor's bound will be presented along with its properties for particular graph families such as t-perfect, h-perfect, and W-perfect graphs. Besides, it will be shown that the derived properties of the improved Shor's bound make it possible single outing a family of graphs for which the weighted independence number can be found in polynomial time.

2. Shor's bounds ψ^* and ψ_1^*. Let us consider a quadratic optimization problem in the following formulation: it is required to find

$$Q_0^* = \sup_{x \in \mathbb{R}^n} Q_0(x) \tag{1}$$

subject to constraints

$$Q_i(x) = 0, \quad i = 1, \ldots, m. \tag{2}$$

Here, $Q_\nu(x)$ are quadratic functions $Q_\nu(x) = (K_\nu x, x) + (b_\nu, x) + c_\nu$, where K_ν are symmetric matrices of order n; $b_\nu \in \mathbb{R}^n$; c_ν are scalars, $\nu = 0, 1, \ldots, m$. Also, some of the quadratic functions $Q_\nu(x), \nu = 0, 1, \ldots, m$ can be linear.

Generally, the problem (1), (2) is multi-extremal and belongs to the class of NP-hard problems. The upper bound for Q_0^* can be obtained as follows. Let $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ be a vector of Lagrange multipliers corresponding to the constraints (2). To the problem (1), (2), the following Lagrangian corresponds:

$$L(x,u) = Q_0(x) + \sum_{i=1}^{m} u_i Q_i(x) = (K(u)x, x) + (b(u), x) + c(u),$$

where

$$K(u) = K_0(x) + \sum_{i=1}^{m} u_i K_i(x), \quad b(u) = b_0 + \sum_{i=1}^{m} u_i b_i, \quad c(u) = c_0 + \sum_{i=1}^{m} u_i c_i.$$

Consider the following function

$$\psi(u) = \sup_{x \in \mathbb{R}^n} L(x,u) = \sup_{x \in \mathbb{R}^n} [(K(u)x, x) + (b(u), x) + c(u)].$$

Let $\Omega^- = \{u : \lambda_{max}[K(u)] < 0\}$ be a subset of those u-components for which $K(u)$ is negative definite matrix, while Ω^0 be a subset of the components for which $\lambda_{max}[K(u)] = 0$. Here, $\lambda_{max}(K(u))$ is the maximum eigenvalue of symmetric matrix $K(u)$ of order n. The domain of function $\psi(u)$ (denoted by $dom\,\psi$) consists of Ω^- and a subset of points $u \in \Omega^0$ for which the next system of equations is compatible:

$$2K(u)x + b(u) = 0. \tag{3}$$

For the remaining points, $\psi(u) = +\infty$. If $dom\,\psi \neq \emptyset$, then there is a nontrivial upper bound for Q_0^*

$$\psi^* = \inf_{u \in dom\,\psi} \psi(u) \tag{4}$$

(the condition $\psi^* = -\infty$ means that the system (2) is incompatible). In this class of Lagrangian bounds, the bound (4) is the optimal (the best) upper bound for Q_0^*. With any predetermined accuracy, the bound ψ^* can be found in polynomial time by methods for minimizing convex non-differentiable functions, such as the ellipsoid method or specific versions of r-algorithms [1]. For instance, an outline of the r-algorithm with an adaptive step control is described in detail in [4].

If ψ^* is attained at $u^* \in \Omega^-$, then

$$\psi^* = \psi(u^*) = Q_0^* = Q_0(x(u^*)),$$

where $x(u^*)$ is a solution of the system (3) for $u = u^*$. Otherwise, ψ^* is attained at the boundary of the domain Ω^-. In this case, there may exist a so-called "duality gap"

$$\Delta^* = \psi^* - Q_0^* > 0.$$

The method of reducing Δ^* proposed by N. Z. Shor is associated with introducing functionally redundant constraints (the set of variables can increase as well), the addition of which leaves many optimal solutions to the problem (1), (2) unchanged. However, the Lagrangian changes in this case, which in some cases can cause reducing the gap between the optimal value Q_0^* of the objective function and the Lagrangian (dual) bound ψ^*. When quadratic functionally redundant constraints

$$Q_{m+1}(x) \leq 0, \ldots, Q_{m+r}(x) \leq 0, r \geq 1,$$

are added to to the original problem (1), (2), then a new quadratic problem is formed: find

$$Q_0^* = \sup_{x \in \mathbb{R}^n} Q_0(x)$$

subject to

$$Q_i(x) = 0,\ i = 1, \ldots, m,$$

$$Q_i(x) \leq 0,\ i = m+1, \ldots, m+r.$$

It corresponds to a longer vector of Lagrange multipliers

$$U = \{\{u\}, u_{m+1}, \ldots, u_{m+r}\}, u_{m+1} \leq 0, \ldots, u_{m+r} \leq 0,$$

and the Lagrangian will have the following form:

$$L_1(x,U) = Q_0(x) + \sum_{i=1}^{m+r} u_i Q_i(x) = L(x,u) + \sum_{i=m+1}^{m+r} u_i Q_i(x).$$

Since $L(x,u) = L_1(x,(\{u\},0,\ldots,0))$, $\psi_1(\{u\},0,\ldots,0) = \psi(u)$, then

$$\psi_1^* = \inf_{U \in dom\,\psi_1} \psi_1(U) \le \inf_{u \in dom\,\psi} \psi(u) = \psi^*, \tag{5}$$

and the functionally redundant constraints can improve Lagrangian bounds.

Note that any constraints, which are linear combinations of existing ones, do not affect the accuracy of the bound ψ_1^*. The contribution of such constraints to the Lagrangian is equivalent only to a certain change in the Lagrange multipliers satisfying the existing constraints. At the same time, the addition of functionally redundant constraints, which are nontrivial consequences of the original problem constraints, in some cases, can result in ψ_1^*, which is more accurate and exact bound on Q_0^* ($\psi_1^* = Q_0^*$).

3. The number $\alpha(G,w)$ and its approximations. Let $G = (V,E)$ be an undirected graph (without loops) with vertex set $V(G)$ and edge set $E(G)$. For each vertex $i \in V(G)$, a positive integer weight w_i is assigned. A subset of vertices $S \subseteq V(G)$ is called a stable (or *independent*) set of graph G, if for any $i,j \in S$, an edge (i,j) does not belong to $E(G)$. The weighted independence number $\alpha(G,w)$ of graph G is defined as $\alpha(G,w) = \max \sum_{i\in S} w_i$, where $S \subseteq V(G)$ is stable set. A subset S^*, where $\alpha(G,w)$ is attained, is called the maximum weighted stable (or *independent*) set of G.

Generally, the problem of finding $\alpha(G,w)$ is NP-hard [2]. Its complexity is easily understood through the general statement of the problem of finding $\alpha(G,w)$ through utilizing a stable set polytope of graph G. Let $\chi^S \in \mathbb{R}^{|V|}$ be an "incident" vector of a vertex subset $S \subseteq V(G)$. This means that $\chi_i^S = 1$ if $i \in S$, and $\chi_i^S = 0$ if $i \in V \backslash S$. A convex hull of "incident" vectors such as χ^S for each stable set S in graph G is called a stable set polytope and is denoted by:

$$STAB(G) := conv\{\chi^S : S \subseteq V(G) - \text{is a stable set}\}.$$

Thus, finding $\alpha(G,w)$ is associated with a maximization of linear function on the convex polytope $STAB(G)$, namely,

$$\alpha(G,w) = \max \sum_{i \in V(G)} w_i x_i, \quad x \in STAB(G). \tag{6}$$

The maximum in the problem (6) is attained at one or several vertices of the polytope $STAB(G)$. Generally, the polytope $STAB(G)$ can possess a highly complex structure, that is why the problem (6) belongs to a class of NP-hard problems.

A polynomial solvability of the problem (6) concerns those graphs whose polytopes $STAB(G)$ possess specific properties. This is due to a very simple fact. Let $LSTAB(G)$ be a polytope given a system of linear inequalities, which is an outer approximation of the polytope $STAB(G)$. Then a solution to the following linear program (LP-program)

$$\alpha_L^*(G,w) = \max \sum_{i \in V(G)} w_i x_i, \quad x \in LSTAB(G) \tag{7}$$

yields a bound $\alpha_L^*(G,w)$ being an upper bound for $\alpha(G,w)$. It is due to the polytope $LSTAB(G)$ is an outer approximation of the polytope $STAB(G)$. Wherefrom, we have $\alpha_L^*(G,w) \geq \alpha(G,w)$. The number of linear inequality-constraints in the LP-program (7) can be exponential. If for a certain family of graphs the polytope $LSTAB(G)$ coincides with the polytope $STAB(G)$, then, for this family, the bound $\alpha_L^*(G,w)$ will be exact, i.e., $\alpha_L^*(G,w) = \alpha(G,w)$. If the bound $\alpha_L^*(G,w)$ can be found in polynomial time, then, for any graph G from this family, the problem of finding $\alpha(G,w)$ is solvable in polynomial time as well.

This principle provides justification of a polynomial solvability of a problem of finding $\alpha(G,w)$ for bipartite, perfect, t-perfect, W-perfect, and some other graphs (see [2], Ch. 9). Each of these graph families posses its own polytopes $LSTAB(G)$ having specific titles in many cases. For example, $FSTAB(G)$ is a fractional stable set polytope, $QSTAB(G)$ is a clique polytope. Also, there exists an odd-cycle polytope, a wheel polytope, etc. Note that, to each of the above graph families, a specific way of finding the bound $\alpha_L^*(G,w)$ is developed. For instance, for a family of perfect graphs $STAB(G) = QSTAB(G)$. For them, finding $\alpha_Q^*(G,w)$ in polynomial time is provided by a well-known Lovasz number $\vartheta(G,w)$ [2]. The bound is a more precise upper bound for $\alpha(G,w)$ than the bound $\alpha_Q^*(G,w)$ for an arbitrary graph G. Similarly, for t-perfect graphs $STAB(G) = CSTAB(G)$. For the family, the polynomial solvability of the problem of finding $\alpha(G,w)$ is caused by the bound $\alpha_C^*(G,w)$ peculiarities. For instance, for graph G from the family, $\alpha(G,w)$ can be found in polynomial time. Note that, to all these polynomially solvable cases, the same meaning can easily be given. For that, the upper bounds for $\alpha(G,w)$ constructed by N. Z. Shor in the scope of study of non-convex quadratic problems can be applied.

4. The bound $\psi(G,w)$ for bipartite and perfect graphs. The simplest upper bound for $\alpha(G,w)$ (further referred to as $\psi(G,w)$) was proposed by N. Z. Shor in [1]. It concerns a statement of the problem of finding $\alpha(G,w)$ as the following non-convex quadratic problem:

$$\alpha(G,w) = \max \sum_{i \in V(G)} w_i x_i \tag{8}$$

subject to

$$x_i x_j = 0 \quad \forall (i,j) \in E(G), \tag{9}$$

$$x_i^2 - x_i = 0 \quad \forall i \in V(G), \tag{10}$$

where the Boolean variable $x_i \in \{0,1\}$ is equal to one, if the vertex i is included in a stable set, otherwise, it is equal to zero. Here, Boolean variables are described by equalities (10). The constraints (9) imply that two vertices cannot simultaneously belong to a stable set if they are linked by an edge in graph G.

In the quadratic problem (8)-(10), the bound $\psi(G,w)$ is the optimal upper bound ψ^* of the objective function maximum in the form (4). With any prescribed accuracy, the bound $\psi(G,w)$ can be found in polynomial time. It was shown (see [1]) that it coincides with the weighted Lovasz number $\vartheta(G,w)$. For an arbitrary graph G, it holds

$$\alpha(G,w) \le \psi(G,w) = \vartheta(G,w).$$

If graph G is perfect or bipartite, then

$$\alpha(G,w) = \psi(G,w) = \vartheta(G,w),$$

and the bound $\psi(G,w)$ is exact for $\alpha(G,w)$. Such a way of justification of the accuracy of the bound $\psi(G,w)$ for these families of graphs is related to the weighted Lovasz numbers' features.

Justification that the simplest Shor's bound coincides with $\alpha(G,w)$ for bipartite and perfect graphs can be made more clear. It is because the quadratic constraints (9)-(10) result in a family of linear inequalities

$$(vertix\ constraints) \quad 0 \le x_i \le 1 \quad \forall i \in V(G), \tag{11}$$

$$(edge\ constraints) \quad x_i + x_j \le 1 \quad \forall (i,j) \in E(G), \tag{12}$$

$$(clique\ constraints) \quad \sum_{i \in V(Q)} x_i \le 1 \quad \forall Q \in G, \tag{13}$$

which are satisfied for the polytope $STAB(G)$. Here, Q is a clique (full subgraph) in graph G. The first set (11) of inequalities are obtained by relaxing (weakening) the constraints (10). The validity of clique inequalities (13) was shown by N. Z. Shor (see [1], p. 252), and the validity of edge inequalities (12) is a consequence of the clique inequalities, if a clique in graph G consists of two vertices, i.e., it coincides with an edge of G.

As a result of the relaxation of the quadratic problem (8)-(10), it is easy to obtain the bound $\alpha^*_F(G,w)$ for the fractional stable set polytope:

$$FSTAB(G) = \{x \in \mathbb{R}^{|V|} : x \text{ satisfies (11) and (12)}\},$$

as well as the bound $\alpha^*_Q(G,w)$ for the click polytope

$$QSTAB(G) = \{x \in \mathbb{R}^{|V|} : x \text{ satisfies (11) and (13)}\}.$$

Since $FSTAB(G) \supseteq QSTAB(G)$, we have a relation:

$$\alpha^*_F(G,w) \ge \alpha^*_Q(G,w) \ge \psi(G,w) \ge \alpha(G,w),$$

wherefrom it follows that the bound $\psi(G,w)$ is exact for both bipartite and perfect graphs. To verify this, let us use that, for bipartite graphs, $\alpha^*_F(G,w) = \alpha(G,w)$ (for them, $STAB(G) = FSTAB(G)$ and there are no isolated vertices); for perfect graphs, $\alpha^*_Q(G,w) = \alpha(G,w)$ (for them, $STAB(G) = QSTAB(G)$).

Note that, for t-perfect and W-perfect graphs, the bound $\psi(G,w)$ will not be exact anymore, but it can be improved by utilizing functionally redundant constraints.

5. The bound $\psi_1(G,w)$ for t-perfect graphs. An improved upper Shor's bound for $\alpha(G,w)$ (further referred to it as $\psi_1(G,w)$) is related to such a non-convex quadratic problem:

$$\alpha(G,w) = \max \sum_{i \in V(G)} w_i x_i \tag{14}$$

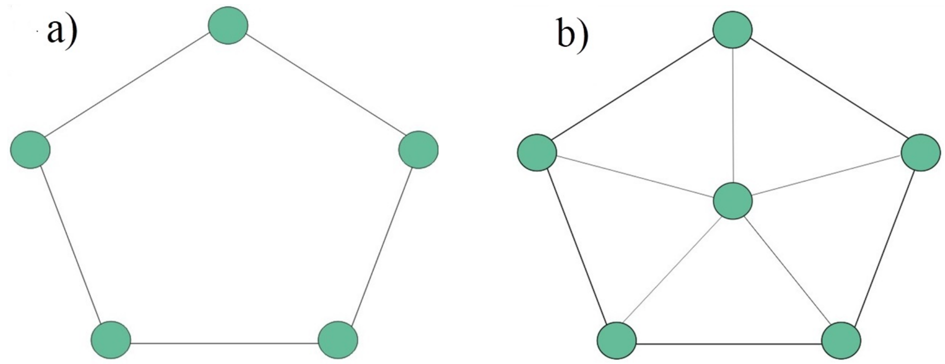

Fig. 1. Odd cycle C_5 and wheel W_6 for $k = 2$

subject to

$$x_i x_j = 0 \quad \forall (i,j) \in E(G), \tag{15}$$

$$x_i^2 - x_i = 0 \quad \forall i \in V(G), \tag{16}$$

$$x_i x_k + x_j x_k \le x_k, \ \forall (i,j) \in E(G), k \ne i, j, \tag{17}$$

The problem (14)-(17) was obtained from (8)-(10) by adding a family of functionally redundant quadratic constraints in the form of inequalities (17), i.e., those that make a set of optimal solutions to the quadratic problem (8)-(10) unchangeable. The functionally redundant constraints are obtained by multiplying edge inequalities (12) by a variable x_k for $k \ne i, j$. A sign of these inequalities remains the same since $x_k = x_k^2 \ge 0$. It is the presence of constraints (17) that gives the bound $\psi_1(G,w)$ various remarkable properties for some families of graphs.

The bound $\psi_1(G,w)$ is the optimal upper bound ψ_1^* of the form (5) for the objective function maximum in the quadratic problem (14)-(17). The bound $\psi_1(G,w)$ can be found with any prescribed accuracy in polynomial time. It is harder to find it than to find $\psi(G,w)$, since the number of Lagrange multipliers is increased because of the presence of functionally redundant constraints (17). What are the advantages of these functionally redundant constraints? First, for an arbitrary graph G, the bound $\psi_1(G,w)$ satisfies a relation

$$\alpha(G,w) \le \psi_1(G,w) \le \psi(G,w),$$

and is always not worse upper bound of $\alpha(G,w)$ than the bound $\psi(G,w)$. Secondly, the bound $\psi_1(G,w)$ is exact for the weighted independence number of t-perfect graphs. This property of the bound $\psi_1(G,w)$ follows from the fact that the next linear inequalities follow from the quadratic constraints (15)-(17):

$$(odd - cycle\ constraints) \qquad \sum_{i \in V(C_{2k+1})} x_i \le k \quad \forall C_{2k+1} \in G, \tag{18}$$

valid for the polytope $STAB(G)$ (see [1], p. 252]). Here, C_{2k+1}, $k = 1, 2, \dots$ is an odd cycle in graph G (i.e., it contains an odd number of vertices). An example of an odd cycle C_5 is shown in Fig. 1.a.

By relaxing the quadratic problem (15)-(17), it is easy to obtain the bound $\alpha_C^*(G,w)$ for the odd cycles' polytope

$$CSTAB(G) = \{x \in \mathbb{R}^{|V|} : x \text{ satisfies } (11), (12) \text{ and } (18)\}.$$

For an arbitrary graph G, the bound satisfies an inequality

$$\alpha^*_C(G,w) \geq \psi_1(G,w) \geq \alpha(G,w).$$

From the latter, it follows that the bound $\psi_1(G,w)$ coincides with $\alpha(G,w)$ for t-perfect graphs, since they all satisfy relations $STAB(G) = CSTAB(G)$ and $\alpha^*_C(G,w) = \alpha(G,w)$.

6. The bound $\psi_1(G,w)$ for W_p-perfect graphs. The question is, will $\psi_1(G,w)$ be an exact bound of $\alpha(G,w)$ for W-perfect graphs? For graphs of the family, the following constraints:

$$(wheel\ constraints) \qquad \sum_{i\in V(C_{2k+1})} x_i + kx_{i_{2k+2}} \leq k, \quad \forall\, W_{2k+2} \in G \tag{19}$$

are valid for the polytope $STAB(G)$. Here, W_{2k+2} is a wheel in graph G (it consists of an odd cycle C_{2k+1} and a vertex associated with each of the cycle vertices). An example of wheel W_6, built based on the odd cycle C_5, is shown in Fig. 1.b. The "wheel"polytope has a form

$$WSTAB(G) = \{x \in \mathbb{R}^{|V|} : x \text{ satisfies } (11),\ (12),\ (18),\ \text{and } (19)\}.$$

It is associated with the bound $\alpha^*_W(G,w)$, which can be found in polynomial time for an arbitrary graph G [2].

It turns out that the bound $\psi_1(G,w)$ coincides with $\alpha(G,w)$ for W-perfect graphs. Moreover, this holds even for a more complex family of graphs than W-perfect ones. This is due to the fact that the constraints (15)-(17) result in linear inequalities

$$(p-wheel\ constraints) \qquad \sum_{i\in V(C_{2k+1})} x_i + k\sum_{j\in V(Q_p)} x_j \leq k, \quad \forall\, W_{2k+1+p} \in G, \tag{20}$$

valid for the polytope $STAB(G)$. Here, the subgraph W_{2k+1+p} is a p-wheel [5]. Vertices of p-wheel W_{2k+1+p} are ones of disjoint odd cycle C_{2k+1} and clique Q_p (a complete subgraph on p-vertices). The edge set of W_{2k+1+p} includes all edges of odd cycle C_{2k+1}, all edges of clique Q_p, as well as edges connecting each vertex C_{2k+1} with all vertices of clique Q_p. Examples of 1-wheel and 2-wheel built based on the odd cycle C_5 are shown in Fig. 2.

Inequalities (20) imply that, for each p-wheel from graph G, either one of the vertices of clique Q_p or k vertices from odd cycle C_{2k+1} can be included. In case, if clique Q_p degenerates into a vertex, from linear inequalities (20) for p-wheel, the linear inequalities (19) for regular wheel W_{2k+2} follow.

By combining the constraints (11)-(13) and (18) with the constraints (20), it is easy to build a generalization of W-perfect graphs for which the Shor's improved bound for $\alpha(G,w)$ is exact. Let a p-wheel polytope be given as

$$W_pSTAB(G) = \{x \in \mathbb{R}^{|V|} : x \text{ satisfies } (11), (12), (18),\ \text{and } (20)\},$$

and it corresponds to an upper bound

$$\alpha^*_{W_p}(G,w) = \max \sum_{i\in V(G)} w_i x_i, \quad x \in W_pSTAB(G).$$

The following theorem holds.

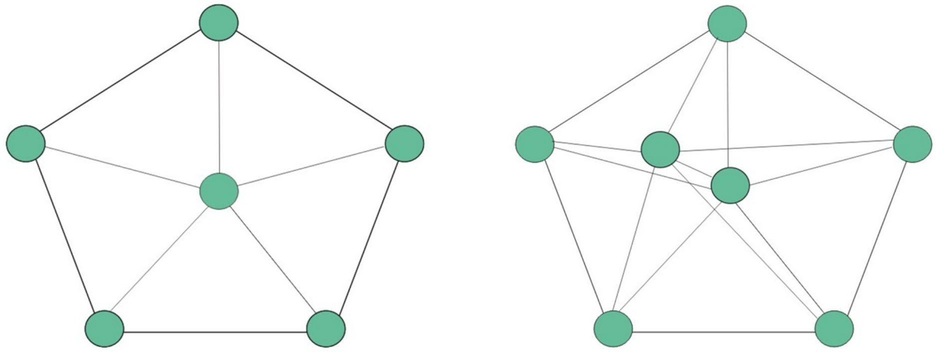

Fig. 2. Examples of 1-wheel and 2-wheel based on odd cycle C_5

Теорема 1. *[3, 6] For an arbitrary graph G, the following two-sided inequality is true:*

$$\alpha^*_{W_p}(G,w) \geq \psi_1(G,w) \geq \alpha(G,w). \tag{21}$$

A family of graphs for which $STAB(G) = W_pSTAB(G)$ is called W_p-perfect. For them, $\alpha^*_{W_p}(G,w) = \alpha(G,w)$, and the inequality (21) turns into equality

$$\alpha^*_{W_p}(G,w) = \psi_1(G,w) = \alpha(G,w), \tag{22}$$

from which the following theorem follows.

Теорема 2. *[3, 6] If graph G is W_p-perfect, then the bound $\psi_1(G,w)$ is equal to $\alpha(G,w)$.*

Since, with any prescribed accuracy, the bound $\psi_1(G,w)$ can be found in polynomial time, from Theorem 2, it follows that, for any graph G in the family of W_p-perfect graphs, the problem of finding $\alpha(G,w)$ is polynomially solvable.

7. Conclusion. The polynomial solvability of the problem of finding $\alpha(G,w)$ for perfect, t- and W-perfect graphs can be justified by relaxing (weakening) the polytope $W_pSTAB(G)$ through removing some of the polytope constraints. For instance, an H-presentation of the polytope $CSTAB(G)$ follows from the one of the polytope $W_pSTAB(G)$ by removing of p-wheel constraints. An H-presentation of the click polytope $QSTAB(G)$ is derived from the one of the polytope $W_pSTAB(G)$ for $k = 1$ (i.e., the only odd cycles coinciding with 3-clique are considered) and p: $1 \leq p \leq |V| - 3$. Here, the inequalities for 2-cliques follow from the edge inequalities (12), the inequalities for 3-cliques – from the inequalities (18) for odd cycles and $k = 1$, and the inequalities for high order cliques – from the inequalities (20) for p-wheels, $k = 1$, and an arbitrary p. The polytope's $WSTAB(G)$ H-presentation is obtained from $W_pSTAB(G)$ for $p = 1$. As a result, an inequality

$$\alpha^*_W(G,w) \geq \psi_1(G,w) \geq \alpha(G,w),$$

holds and results in coincidence the bound $\psi_1(G,w)$ with $\alpha(G,w)$, if graph G is W-perfect (here, $\alpha^*_W(G,w) = \alpha(G,w)$).

In conclusion, we indicate that, by the Shor's bound $\psi^*_1(G,w)$, it can be justified a polynomial solvability of the problem of finding the weighted independence number for a more complex families of graphs than W_p-perfect ones. Also, polynomial solvability of combinatorial optimization problems has been proven for 2-level polytopes [8] formed a wide class of polytopes related to graphs problems and other ones. It is highly promising extending the results of the paper and combining them with approaches presented in [9] onto the whole class of 2-level and multilevel polytopes [8]. Moreover, one can construct many new families of linear inequalities that are valid for the polytope $STAB(G)$ and characterize more complex substructures in graph obtained by a certain combination of cliques Q_p, odd cycles C_{2k+1}, and their complements $\overline{C}_{2k+1}$. By the bound $\psi_1(G, w)$, this makes it possible to single out new families of graphs for which the problem (6) is polynomially solvable. Moreover, based on applying non-differentiable optimization methods, it allows finding upper bounds of the complexity of these classes of problems. As a result, this can be a great contribution to the available results (see [2, 7]) and, likely, for many new families of graphs, it will make possible finding a general form of describing the external approximation polytope $LSTAB(G)$ for the polytope $STAB(G)$.

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Стецюк П. І., Пічугіна О. С. Оцінки Шора для зваженого числа стійкості графа.

Описано застосування техніки двоїстих лагранжевих квадратичних оцінок Н. З. Шора до дослідження задачі про максимальну зважену незалежну множину вершин графа. Наведено отримані за її допомогою дві верхні оцінки Н. З. Шора для $\alpha(G, w)$ – максимального зваженого числа стійкості графа, які можна знайти за поліноміальний час. Перша оцінка $\psi(G, w)$ пов'язана з квадратичною моделлю задачі про максимальну зважену незалежну множину вершин графа та співпадає з відомим числом Ловаса $\vartheta(G, w)$. Друга оцінка $\psi_1(G, w)$ відповідає цій же квадратичній моделі, яка доповнена сімейством функціонально надлишкових квадратичних обмежень, та спроможна покращити точність верхньої оцінки $\alpha(G, w)$ у спеціальних сімействах графів. Показано, що $\psi(G, w) = \alpha(G, w)$, якщо граф є дводольним або досконалим, а

$\psi_1(G, w) = \alpha(G, w)$, якщо граф є t- або W_p-досконалим. На основі цих виділених класів графів продемонстровано технологію формування нових класів графів, для яких зберігається поліноміальна розв'язність задачі знаходження максимального зваженого числа стійкості графа. Таким чином, на прикладі задачі про максимальну зважену незалежну множину вершин графа показано, яким чином техніка лагранжевих оцінок може бути застосована до вирішення проблеми виділення класів поліноміально розв'язних задач комбінаторної оптимізації. Дана концепція може бути використана як для уточнення існуючих оцінок цільової функції в задачах комбінаторної оптимізації, так і для обґрунтування їх поліноміальної розв'язності.

Ключові слова: Задача про максимальну зважену незалежну множину графа, зважене число стійкості графа, квадратична оптимізація, двоїсті лагранжеві оцінки, поліедральна релаксація, досконалий граф, дводольний граф.

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