UDC 512.628.2
DOI https://doi.org/10.24144/2616-7700.2021.39(2).60-67

I. O. Melnyk[1], R. V. Kolyada[2], O. M. Melnyk[3]

[1] Ivan Franko National University of Lviv,
Associate professor of the department of algebra, topology and fundamentals of mathematics,
Candidate of physical and mathematical sciences
ivannamelnyk@yahoo.com
ORCID: https://orcid.org/0000-0002-7650-5190

[2] Ukrainian Academy of Printing,
Associate professor of the department of applied mathematics and physics,
Candidate of physical and mathematical sciences
rostyslavakolyada@gmail.com
ORCID: https://orcid.org/0000-0003-4371-070X

[3] National University "Lviv Polytechnic", Ukrainian Academy of Printing,
Associate professor of the department of computational mathematics and programming, associate professor of the department of applied mathematics and physics,
Candidate of physical and mathematical sciences
melnykorest@gmail.com
ORCID: https://orcid.org/0000-0003-4456-4759

SOME PROPERTIES OF DIFFERENTIAL, QUASI-PRIME AND DIFFERENTIALLY PRIME SUBSEMIMODULES

The notion of a semiring derivation is traditionally defined as an additive map satisfying the Leibnitz rule, i. e. a map $\delta\colon R \to R$ is called a derivation on R if $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)b + a\delta(b)$ for any $a, b \in R$. The notion of a `quasi-prime ideal`, for the first time, was introduced in differential commutative rings, i.e. commutative rings considered together with a derivation, as a differential ideal maximal among those disjoint from some multiplicatively closed subset of a ring. A subsemimodule P of a semimodule M is called prime if for any ideal I of R and any subsemimodule N of M the inclusion $IN \subseteq P$ follows $N \subseteq P$ or $I \subseteq (P : M)$. A differential subsemimodule P of M is called a `differentially prime subsemimodule` if for any $r \in R$, $m \in M$, $k \in \mathbb{N}_0$, $rm^{(k)} \in P$ follows $r \in (P : M)$ or $m \in P$.

The present paper is devoted to investigating the notions of `differential subsemimodule`, `differentially prime subsemimodule`, and `quasi-prime subsemimodule` of a differential semimodule (which is defined as a semimodule together with a derivation on it related to the corresponding semiring derivation), not necessarily commutative. The objective of the article is to investigate some properties of such subsemimodules, and to show the interrelation between `quasi-prime ideals` and `differentially prime subsemimodule` in case of differential semimodules satisfying the ascending chain condition for differential subsemimodules. The paper consists of two main parts. In the first part, the author investigates some properties of differential subsemimodules and the corresponding differential ideals, and gives some examples of such subsemimodules. The second part of the paper is devoted to considering the connection existing between `quasi-prime subsemimodules` and `differentially prime subsemimodules`. It is established that a differential subsemimodule N of M is differentially prime if and only if N is a quasi-prime subsemimodule for a differential semimodule M satisfying the ascending chain condition for differential subsemimodules.

Keywords: semimodule derivation, semiring derivation, differential semimodule, differential semiring, differential ideal, differential subsemimodule, differentially prime subsemimodule, quasi-prime subsemimodule.

1. Introduction. The notion of a derivation for semirings is defined in [1] as an additive map satisfying the Leibnitz rule. Recently in [2], [3], [4] the authors investigated different properties of semiring derivations, differential semirings, i.e. semirings considered together with a derivation, and differential ideals of such rings. Prime subsemimodules of semimodules over semirings were introduced and studied in [5]. Differentially prime ideals were introduced in [6] for differential, not necessarily commutative, rings. Differentially prime submodules of modules over associative rings were studied in [7], [8]. Quasi-prime ideals of differential rings were introduced and studied in [9], [10], its generalizations to differential modules, semirings and semimodules were studied by different authors, e.g. [3], [4].

Due to the development of semiring and semimodule theory recently, the need of studying the properties of differential semirings, differential semimodules, semiring ideals and subsemimodules defined by similar conditions arose. The objective of this paper is to introduce and investigate differential semimodules over differential semirings. We extend some basic results on differential modules to differential semimodules over differential semirings.

2. Preliminaries. For the sake of completeness some definitions and properties used in the paper will be given here. For more information see [1], [5]. Throughout the paper $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$ the set of non-negative integers.

A semiring is a non-empty set R equipped with two associative binary operations called addition (denoted by $+$) and multiplication (denoted by $\cdot$), such that multiplication distributes over addition from either side. A semiring which is not a ring is called a *proper semiring*. A semiring $(R, +, \cdot)$ is said to be *commutative* if $\cdot$ is commutative on R.

Zero $0_R \in R$ is called (*multiplicatively*) *absorbing* if $a \cdot 0_R = 0_R \cdot a = 0$ for all $a \in R$. An element $1_R \in R$ is called *identity* if $a \cdot 1_R = 1_R \cdot a = a$ for all $a \in R$. Suppose $1_R \neq 0_R$.

A subset S of R closed under addition and multiplication is called a *subsemiring* of R. A *semifield* is a semiring in which non-zero elements form a group under multiplication.

The *center* of a semiring R is a set $Z(R) = \{r \in R | rs = sr \ \forall s \in R\}$. It is a subsemiring of R. Since $0 \in Z(R)$, $Z(R) \neq \varnothing$. An element $r \in Z(R)$ is called *central.* A semiring R is commutative if $Z(R) = R$.

A *left ideal* of a semiring R is a nonempty set $I \neq R$ which is closed under $+$ and satisfying the following conditions $ra \in I$ for all $a \in I$, $r \in R$. Similarly we can define right ideal and two-sided ideal of a semiring.

An ideal I of a semiring R is called *subtractive* (or *k-ideal*) if $a \in I$ and $a + b \in I$ follow $b \in I$.

Let R be a semiring with $1_R \neq 0_R$. A *left semimodule over a semiring* R (or *R-semimodule*) is a nonempty set M together with two operations $+\colon M \times M \to M$ and $\cdot\colon R \times M \to M$ such that $(M, +)$ is a commutative monoid with 0_M, $(M, \cdot)$ is a semigroup, $(r+s)m = rm+sm$ for all $r, s \in R$, $m \in M$, $r(m_1+m_2) = rm_1+rm_2$ for all $r \in R$, $m_1, m_2 \in M$, $0_R \cdot m = r \cdot 0_M = 0_M$ for all $r \in R$ and $m \in M$, $1_R \cdot m = m$ for all $m \in M$. An R-subsemimodule M is called a *semivector space* if R is a semifield. A subset N of an R-semimodule M is called a *subsemimodule* of M if N itself is a semimodule with respect to the operations for M, i. e. if $m + n \in N$ and $rm \in N$ for any $m, n \in N$, and $r \in R$. A subsemimodule N of an R-semimodule M is called *subtractive* or *k-subsemimodule* if $m_1 \in N$ and $m_1 + m_2 \in N$ follow $m_2 \in N$. So

$\{0_M\}$ is a subtractive subsemimodule of M.

3. Differential semimodules and subsemimodules. Let R be a semiring. A map $\delta\colon R \to R$ is called a *derivation on* R [1] if $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)b + a\delta(b)$ for any $a, b \in R$. A semiring R equipped with a derivation δ is called a *differential semiring* with respect to the derivation δ (or *δ-semiring*), and denoted by (R, δ) [2]. A derivation of a semiring R is called *trivial* if it sends all a in R to 0_R. A semiring is called *differentially trivial* if it has no non-trivial derivation.

Let M be a left semimodule over the semiring R. A map $d\colon M \to M$ is called a *derivation* of the semimodule M, associated with the semiring derivation $\delta\colon R \to R$ (or a δ-derivation) if the following conditions hold:

1) $d(m+n) = d(m) + d(n)$ for any $m, n \in M$;
2) $d(rm) = \delta(r)m + rd(m)$ for any $m \in M$, $r \in R$.

Since the semiring $\mathbb{N}_0$ is differentially trivial, any derivation of $\mathbb{N}_0$-semimodule is a semimodule homomorphism. In a $\mathbb{N}_0$-semimodule any semimodule homomorphism is a semimodule derivation associated with the trivial derivation on $\mathbb{N}_0$.

The same is true for a semimodule over the differentially trivial semiring $\mathbb{Q}_+$.

A left R-semimodule M together with a derivation $d\colon M \to M$ is called a *differential semimodule* (or *d-δ-semimodule*) and denoted by (M, d).

Zero semimodule (0_M) is a differential semimodule over any differential semiring under any semimodule derivation. Any semiring can be considered as a differential $\mathbb{N}_0$-semimodule under derivation associated with the trivial derivation on $\mathbb{N}_0$. Here any semimodule homomorphism is a semimodule derivation. Any differential left ideal of the differential semiring R is a left differential semimodule over R. Any differential semiring is a left (right) differential semimodule over itself. Any differential module over a differential ring R is a differential semimodule over R as a differential semiring. Any differential semivector space over a differential semifield F is a differential semimodule over a differential semiring F. If R is a semiring and $A \neq \varnothing$ is any set then the set R^A of all differentiable functions from A to R is a left differential R-semimodule, where the addition and scalar multiplications is defined elementwise and the derivation is defined in the ordinary way.

Proposition 1. *Let (R, δ) be a differential semiring and let $\{(M_i, d_i) | i \in I\}$ be a family of left differential semimodules over R, where all the semimodule derivations $d_i\colon M_i \to M_i$ are associated with the semiring derivation δ.*

A direct product $\prod_{i\in I} M_i$ is a left differential R-semimodule.

Proof. As shown in [1], $\prod_{i\in I} M_i$ has a structure of a left R-semimodule under componentwise addition and scalar multiplication.

Define a map $d\colon \prod_{i\in I} M_i \to \prod_{i\in I} M_i$ by the rule $d\colon ((m_i)) \mapsto (d_i(m_i))$. It is easy so see that d is additive. Now consider $d(r \cdot (m_i)) = (d_i(rm_i)) = (\delta(r)m_i + rd_i(m_i)) = = \delta(r)(m_i) + r(d_i(m_i)) = \delta(r)(m_i) + r \cdot d(m_i)$. Hence M is a differential δ-semimodule over R.

If not stated otherwise, in what follows let (R, δ) be a differential semiring, (M, d) be a left differential semimodule over R.

A subsemimodule N of the R-semimodule M is called *differential* if $d(m) \in N$ whenever $m \in N$. Any differential semimodule has two trivial differential subsemimodules: $\{0_M\}$ and itself.

An element $m \in M$ is called *additively idempotent* if $m + m = m$. Denote by $I^+(M)$ the set of all additively idempotent elements of M. A semimodule M is called *additively idempotent* if every element of M is additively idempotent, i. e. $I^+(M) = M$. It is easy to see that every semimodule over an additively idempotent semiring is additively idempotent.

Proposition 2. *The set $I^+(M)$ of all additively idempotent elements of M is a differential subsemimodule of M.*

Proof. Straightforward.

According to [1], the *zeroid* of M is defined to be $Z(M) = \{m \in M | m + x = x$ for some $x \in M\}$. $Z(M)$ is a subtractive subsemimodule of M containing $I^+(M)$. Moreover $Z(R)M \subseteq M$.

Proposition 3. *$Z(M)$ is a differential subsemimodule of M.*

Proof. Straightforward.

Proposition 4. *Let $\{N_\alpha\}_{\alpha\in A}$ be a family of subsemimodules of M.*

1) *An intersection $\bigcap_{\alpha\in A} N_\alpha$ of any family of differential (subtractive) subsemimodules of M is a differential (subtractive) subsemimodules of M.*
2) *A sum $\sum_{\alpha\in A} N_\alpha$ of any family of differential subsemimodules of the differential semimodule M is a differential subsemimodules of M.*

Proof. Straightforward.

Proposition 5. *If I is a left differential ideal of R and N is a differential R-subsemimodule of M, then set IN consisting of all finite sums of elements $r_i m_i$ with $r_i \in R$ and $m_i \in M$, is a differential R-subsemimodule of M.*

Proof. Suppose $x \in IN$. Then $x = \sum_{i=1}^{k} r_i m_i$ for some $m_i \in N$, $r_i \in I$, $i \in \{1, 2, \dots, k\}$, $k \in \mathbb{N}_0$. We have that

$$d(x) = \sum_{i=1}^{k} \left(\delta(r_i) m_i + r_i d(m_i)\right) = \sum_{i=1}^{k} \delta(r_i) m_i + \sum_{i=1}^{k} r_i d(m_i).$$

Since I and N are differential, then $\delta(r_i) \in I$ and $d(m_i) \in N$ for all $i \in \{1, 2, \dots, n\}$. Hence $d(x) \in IN$.

Corollary 1. *If I is a left differential ideal of the differential semiring R, then the set IM is a differential subsemimodule of the left differential R-semimodule M.*

Proposition 6. *If I is a differential ideal of R and N is a differential subtractive subsemimodule of M, then the residual $(N : I) = \{m \in M | Im \subseteq N\}$ is a differential subtractive R-subsemimodule of M.*

Proof. It is easy to prove that $(N : I)$ is a subtractive subsemimodule of M. Let $x, x + y \in (N : I)$. Then for every $a \in I$, $ax \in N$ and $a(x + y) \in N$. By subtractiveness of N the latter implies $ay \in N$. So $y \in (N : I)$.

Let $m \in (N : I)$. Then $am \in N$ for every $a \in I$. Since N is differential $d(am) = \delta(a) m + a d(m) \in N$. The differentiality of I follows $\delta(a) m \in N$. Given N is subtractive subsemimodule, we have $a d(m) \in N$. Hence $d(m) \in (N : I)$.

Corollary 2. $(0 : I) = \{m \in M | Im = \{0_M\}\}$ *is a differential subtractive subsemimodule of M.*

Proposition 7. *If N is a differential subtractive subsemimodule of M and X is a non-empty differentially closed subset of M, then $(N : X) = \{r \in R | rX \subseteq N\}$ is a differential subtractive left ideal of R.*

Proof. The residual $(N : X)$ is a subtractive left ideal of R by [1], Proposition 14.24. Let $r \in (N : X)$. Then $d(rx) = \delta(r)x + rd(x) \in N$ for every $x \in X$. Since X is differentially closed, $rd(x) \in N$. Given N is a subtractive subsemimodule, $\delta(r) \in (N : X)$.

Corollary 3. *Let N, K be differential subtractive subsemimodules of the differential left R-semimodule M. Then $(N : K)$, $(N : M)$, $(0 : M)$ are differential subtractive left ideals of R.*

An annihilator of an R-semimodule M is a set $Ann(M) = (0 : M)$. *An annihilator of the element $m \in M$* is a set $Ann(m) = (0 : Rm) = \{r \in R | rm = 0\}$. In [5] it is shown that $Ann(M) = (0 : M)$, $(N : m) = \{r \in R | rm \in N\}$ and $Ann(m) = (0 : m)$ are subtractive ideals of R, where R is a commutative semiring. For a differential semimodule M $Ann(M)$ is always a differential ideal, however $Ann(m)$ is generally not. So we define a differential analogue of the annihilator of an element $m \in M$.

For an element $m \in M$ denote by $m^{(0)} = m$, $m' = d(m)$, $m'' = d(m')$, $m^{(n)} = d\left(m^{(n-1)}\right)$, $n \in \mathbb{N}_0$. Moreover, let $m^{(\infty)} = \{m^{(n)} | n \in \mathbb{N}_0\}$. It is easy to see that the set $m^{(\infty)}$ is differentially closed. A *differential annihilator of the element $m \in M$* is the set $Ann_d(m) = \left(0 : m^{(\infty)}\right) = \{r \in R | rm^{(\infty)} = 0_M\}$. By Proposition 7 we have the following corollary

Corollary 4. *If N is a differential subtractive subsemimodule of M, then for any $m \in M$ the set $(N : m^{(\infty)}) = \{r \in R | rm^{(\infty)} \subseteq N\}$ is a differential subtractive left ideal of R.*

Corollary 5. *If $m \in M$ then $Ann_d(m) = \left(0 : m^{(\infty)}\right)$ is a differential subtractive left ideal of R. Moreover, $Ann_d(m) \subseteq Ann(m)$.*

Proposition 8. *Let T be a non-empty subset of M. An intersection of all differential subtractive subsemimodules of the differential R-semimodule M containing the set T is a differential subtractive subsemimodule of M. It is the smallest differential subsemimodule of M containing T.*

Proof. It is obvious that $\bigcap_{T \subseteq N} N$ is a subsemimodule of M. Clearly it is subtractive. Let $m \in \bigcap_{T \subseteq N} N$. Then m is contained in any subsemimodule containing T. Therefore, $d(m) \in \bigcap_{T \subseteq N} N$.

Denote by $(T) = \bigcap_{T \subseteq N} N$ the subsemimodule generated by the set T. A subsemimodule (T) is called *finitely generated* if the set T is finite.

Denote $[T] = \bigcap_{T \subseteq N} N$ and call it the *subsemimodule differentially generated by the set T.*

Proposition 9. *A differential subsemimodule $[T]$ is generated by the set $\bigcap_{k \in \mathbb{N}_0} d^{(k)}(T)$ as an R-subsemimodule.*

Proof. Straightforward.

Example. For an $\mathbb{N}_0$-subsemimodule $\mathbb{N}_0[x]$ under the ordinary derivation $\delta = \frac{d}{dx}$, where $\delta(x) = 1$, consider a differential subsemimodule $N = [x^2]$. By Proposition 8, $N = [x^2] = (x^2, 2x, 2) = (x^2, 2)$.

According to [1], a map $f\colon M_1 \to M_2$ is called a *homomorphism* of R-semimodules if $f(x+y) = f(x)+f(y)$ and $f(rx) = rf(x)$ for all $x, y \in M$ and $r \in R$. The *kernel* of f is defined to be the set $Ker(f) = \{m \in M | f(m) = 0_{M_2}\}$, and the image of f is the set $Im(f) = \{f(m) \in M_2 | m \in M_1\}$. $Ker(f)$ is a subtractive subsemimodule of M_1 and $Im(f)$ is a subsemimodule of M_2.

Let (R, δ) be a differential semiring. Let (M_1, d), (M_1, d) be left differential semimodules over R. A homomorphism of differential R-semimodules $f\colon M_1 \to M_2$ is called a *differential R-homomorphism* if $f(d(m)) = d(f(m))$ for all $m \in M_1$.

Proposition 10. *Let (R, δ) be a differential semiring, (M_1, d) and (M_2, d) be differential semimodules over R, and $f\colon M_1 \to M_2$ be a differential R-homomorphism. Then*

1) *$Kerf$ is a differential subtractive subsemimodule of M_1;*
2) *Imf is a differential subsemimodule of M_2;*
3) *If N is a differential subsemimodule of M_1, then N^e is a differential subsemimodule of M_2;*
4) *If N is a differential subtractive ideal of M_2, then N^c is a differential subtractive ideal of M_1.*

Proof. Straightforward.

Corollary 6. *Let M_1 and M_2 be differential semimodules. If $f\colon M_1 \to M_2$ is a differential semimodule homomorphism and N is a prime differential subtractive subsemimodule of M_2, then $f^{-1}(N)$ is a prime differential subtractive subsemimodule of M_1.*

As a consequence we obtain the following

Theorem 1. *Let M_1 and M_2 be differential semirings, and let $f\colon M_1 \to M_2$ be a differential semiring homomorphism. Then f induces a differential isomorphism $\bar{f}\colon M_1/Kerf \to Imf$ for which $\bar{f}(m + Kerf) = f(m)$ for all $m \in M_1$.*

4. Quasi-prime and differentially prime subsemimodules. In what follows semirings are considered to be commutative.

A differential subsemimodule N of the left differential semimodule M is called *quasi-prime* if it is maximal differential subsemimodule of M disjoint from some S-closed subset of M. A differential subsemimodule P of M is called *differentially prime* if for any $r \in R$, $m \in M$, $k \in \mathbb{N}_0$, $rm^{(k)} \in P$ follows $r \in (P : M)$ or $m \in P$.

Proposition 11. *If P is a differentially prime subtractive subsemimodule of M, then $(P : M)$ is a differentially prime subtractive ideal of R.*

Proof. $(P : M)$ is subtractive [4]. Let P be differentially prime subsemimodule of M, and let $a^{(k)}b^{(l)} \in (P : M)$ for some $a, b \in R$, $k, l \in \mathbb{N}_0$. Then $a^{(k)}b^{(l)}m \in P$ for all $m \in M$. Therefore, $a^{(k)}(b^{(l)}m) = a^{(k)}(bm)^{(s)} \in P$ follows $a \in (P : M)$ or $bm \in M$, by differential primeness of P. Hence, $b \in (P : M)$.

Theorem 2. *Let M be a differential semimodule satisfying the ascending chain condition for differential subsemimodules. A differential subsemimodule N of M is differentially prime if and only if N is a quasi-prime subsemimodule.*

Proof. Let N be a quasi-prime subsemimodule of M. Suppose there exist $r \in R$, $m \in M$ such that $[r] \cdot [m] \subseteq N$, $r \in R \setminus (N : M)$ and $m \in M \setminus N$. It is clear that $N \subset N + [m]$ and $(N : M) \subset (N : M) + [r]$ and $(N : M) + [r]$ is a differential ideal of R, $N+[m]$ is a differential submodule of M. Since N is a maximal differential submodule satisfying $N \cap X = \varnothing$ for some S-closed subset X of M, for the differential ideal $(N : M) + [r]$ and the differential subsemimodule $N + [m]$ we have that $((N : P) + [r]) \cap S \neq \varnothing$ and $(N + [m]) \cap X \neq \varnothing$. Then $s \in (N : M) + [r]$ and $x \in N + [m]$ for $s \in S$, $x \in X$. Therefore, $sx^{(n)} \in X$ for some $n \in \mathbb{N}_0$, because X is an S-closed subset of M. Then $sx^{(n)} \in ((N : M) + [r]) \cdot (N + [m]) = = (N : M)N+(N : M)\cdot[m]+[r]\cdot N+[r]\cdot[m] \subseteq N$. It follows that $sx^{(n)} \in X \bigcap N \neq \varnothing$, which contradicts to the fact that X is disjoint from N. Hence, N is differentially prime.

If N is a differentially prime subsemimodule of M, then $X = M \backslash N$ is a Sdm-system for some dm-system S of R. Since N is maximal differential subsemimodule not meeting $X = M \backslash N$, then it is quasi-prime.

5. Conclusions. In this article we study differential subsemimodules, quasi-prime and differentially prime subsemimodules. Namely, we give new examples of such subsemimodules, prove some of their properties. We also describe the interrelation between quasi-prime and differentially prime subsemimodules. The obtained results can be used in further study of differential semimodules.

References

1. Golan, J. S. (1999). Semirings and their Applications. *Kluwer Academic Publishers.*
2. Chandramouleeswaran, M., & Thiruveni, V. (2010). On derivations of semirings. *Advances in Algebra, 1,* 123–131.
3. Melnyk, I. (2016). On the radical of a differential semiring ideal. *Visnyk of the Lviv. Univ. Series Mech. Math., 82,* 163-173.
4. Melnyk, I. (2020). On quasi-prime differential semiring ideals. *Scientific Bulletin of Uzhhorod University. Series of mathematics and informatics, 2*(37), 75-81. https://doi.org/10.24144/2616-7700.2020.2(37).75-81.
5. Atani, R. E., & Atani, S. E. (2010). On subsemimodules of semimodules. *Bul. Acad. Stiinte Repub. Moldova. Matematica, 2*(63). 20-30.
6. Khadjiev, Dj., & Çallıalp, F. (1996). On a differential analog of the prime-radical and properties of the lattice of the radical differential ideals in associative differential rings. *Tr. J. of Math., 4*(20), 571-582.
7. Melnyk, I. (2008). *Sdm*-systems, differentially prime and differentially primary modules. *Scientific Bulletin of Uzhhorod University. Series of mathematics and informatics, 16,* 110-118.
8. Melnyk, I.(2008). Differentially prime, quasi-prime and $\Delta - MP$–modules. *Bul. Acad. Stiinte Repub. Moldova. Matematica, 3*(58), 112-115.
9. Keigher, W. (1977). Prime differential ideals in differential rings. *Contributions to Algebra, A Collection of Papers Dedicated to Ellis Kolchin, Academic Press*, 239-249.
10. Keigher, W. F. (1978). Quasi-prime ideals in differential rings. *Houston J. Math., 4*(3), 379-388.

Мельник І. О., Коляда Р. В., Мельник О. М. Деякі властивості диференціальних, квазіпервинних та диференціально-первинних піднапівмодулів

Поняття **диференціювання напівкільця** традиційно визначають як адитивне відображення, яке задовольняє правило Лейбніца, тобто відображення $\delta \colon R \to R$ називають диференціюванням напівкільця R, якщо $\delta(a+b) = \delta(a) + \delta(b)$ і $\delta(ab) = \delta(a)b + a\delta(b)$ для будь-яких $a, b \in R$. Поняття **квазіпервинний ідеал** було вперше введено в комутативних диференціальних кільцях, тобто комутативних кільцях, які розглядаються разом із заданим на них диференціюванням, як диференціальний ідеал, максимальний серед диференціальних ідеалів, які не перетинаються із деякою мультиплікативно-замкненою підмножиною кільця. Піднапівмодуль P напівмодуля M називають первинним, якщо для будь-якого ідеалу I напівкільця R та будь-якого піднапівмодуля N напівмодуля M з $IN \subseteq P$ випливає $N \subseteq P$ або $I \subseteq (P : M)$. Диференціальний піднапівмодуль P напівмодуля M називають диференціально-первинний піднапівмодуль, якщо для будь-яких $r \in R$, $m \in M$, $k \in \mathbb{N}_0$ з $rm^{(k)} \in P$ випливає, що $r \in (P : M)$ або $m \in P$.

Ця стаття присвячена дослідженню понять диференціальний піднапівмодуль, диференціально-первинний піднапівмодуль, квазіпервинний піднапівмодуль в диференціальних напівмодулях (які означаються як напівмодулі разом із диференціюванням, заданому на них, яке узгоджується з відповідним диференціюванням напівкільця). Метою статті є дослідити деякі властивості таких піднапівмодулів, показати взаємозв'язки між квазіпервинними піднапівмодулями та диференціально-первинними піднапівмодулями у випадку диференціальних напівмодулів, що задовольняють умову обриву зростаючих ланцюгів диференціальних піднапівмодулів. Стаття складається з двох основних частин. У першій частині автор досліджує деякі властивості диференціальних піднапівмодулів та відоповідних диференціальних ідеалів, а також наводить деякі приклади таких піднапівмодулів. У другій частині статті розглядаються ланцюги зв'язки, що існують між поняттями квазіпервинний піднапівмодуль та диференціально-первинний піднапівмодуль. Встановлено, що диференціальний піднапівмодуль N напівмодуля M є диференціально-первинний піднапівмодуль тоді і тільки тоді, коли N є квазіпервинний піднапівмодуль диференціального напівмодуля M, який задовольняє умову обриву зростаючих ланцюгів диференціальних піднапівмодулів.

Ключові слова: диференціювання напівмодуля, диференціювання напівкільця, диференціальний напівмодуль, диференціальне напівкільце, диференціальний ідеал, диференціальний піднапівмодуль, диференціально-первинний піднапівмодуль, квазіпервинний піднапівмодуль.

Список використаної літератури

1. Golan J. S. Semirings and their Applications. *Kluwer Academic Publishers*, 1999.
2. Chandramouleeswaran M., Thiruveni V. On derivations of semirings. *Advances in Algebra*, 2010. No. 1. P. 123-131.
3. Melnyk I. On the radical of a differential semiring ideal. *Visnyk of the Lviv. Univ. Series Mech. Math.* 2016. No. 82. P. 163-173.
4. Melnyk I. On quasi-prime differential semiring ideals. *Scientific Bulletin of Uzhhorod University. Series of mathematics and informatics*, 2020. No. 2(37). P. 75-81. DOI: https://doi.org/10.24144/2616-7700.2020.2(37).75-81.
5. Atani R. E., Atani S. E. On subsemimodules of semimodules. *Bul. Acad. Stiinte Repub. Moldova. Matematica*, 2010. No. 2(63). P. 20-30.
6. Khadjiev Dj., Çallıalp F. On a differential analog of the prime-radical and properties of the lattice of the radical differential ideals in associative differential rings. *Tr. J. of Math.*, 1996. No. 4(20). P. 571-582.
7. Melnyk I. *Sdm*-systems, differentially prime and differentially primary modules. *Scientific Bulletin of Uzhhorod University. Series of mathematics and informatics* 2008. No. 16. P. 110-118.
8. Melnyk I. Differentially prime, quasi-prime and $\Delta - MP$–modules. *Bul. Acad. Stiinte Repub. Moldova. Matematica*, 2008. No. 3(58). P. 112-115.
9. Keigher W. Prime differential ideals in differential rings. *Contributions to Algebra, A Collection of Papers Dedicated to Ellis Kolchin, Academic Press*, 1977. P. 239-249.
10. Keigher W. F. Quasi-prime ideals in differential rings. *Houston J. Math.*, 1978. No. 4(3). P. 379-388.

Recived 17.09.2021