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ON DIFFERENTIALLY PRIME IDEALS OF NOETHERIAN SEMIRINGS

The paper is devoted to the investigation of the notion of a `differentially prime ideal` of a differential commutative semiring (i. e. a semiring equipped with a derivation), and its interrelation with the notions of a `quasi-prime ideal` and a `primary ideal`. The notion of a semiring derivation is traditionally defined as an additive map satisfying the Leibnitz rule, i. e. a map $\delta\colon R \to R$ is called a derivation on R if $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)b + a\delta(b)$ for any $a, b \in R$.

A differential ideal P of R is called a `differentially prime ideal` if for any $a, b \in R$, $k \in \mathbb{N}_0$, $ab^{(k)} \in P$ follows $a \in P$ or $b \in P$. It is proved that an ideal P of a semiring R is prime if and only if for any ideals I and J of R the inclusion $IJ \subseteq P$ follows $I \subseteq P$ or $J \subseteq P$. A `quasi-prime ideal` is a differential ideal of a semiring which is maximal among those ideals disjoint from some multiplicatively closed subset of a semiring.

In this paper we investigate some properties of such differentially prime ideals, in particular in case of differential Noetherian semirings. The paper consists of two main parts. The first part of the paper is devoted to establishing some properties of differentially prime ideals and gives some examples of such ideals. In the second part, the author investigates the connection existing between `quasi-prime ideals`, `primary ideals` and `differentially prime ideals` in differential `Noetherian semirings`. It is established that in a differential Noetherian semiring R a differential ideal I of R is differentially prime if and only if I is a quasi-prime ideal.

Keywords: semiring derivation, differential semiring, differential semiring ideal, differentially prime ideal, quasi-prime ideal, primary ideal, Noetherian semiring.

1. Introduction. The notion of a derivation for semirings, defined in [1] as an additive map satisfying the Leibnitz rule, as well as the notion of a differential semiring recently received a lot of attention in [2–4]. Different properties of semiring derivations, differential semirings, ideals of differential semirings were investigated. Quasi-prime ideals of differential rings were introduced and studied in [5,6], its generalizations for differential modules, semirings and semimodules were studied in [3, 4, 7, 8]. Differentially prime ideals were introduced in [9] for differential, not necessarily commutative, rings. Differentially prime submodules of modules over associative rings were studied in [10]. In [11] the author studied the relations between differentially prime and quasi-prime submodules of differential modules. The objective of this paper is to investigate some properties of differentially prime ideals of differential commutative semirings. We aim to find the connections between differentially prime ideals and other types of ideals of differential semirings with the ascending chain condition for their ideals (Noetherian semirings), in particular quasi-prime ideals.

For the sake of completeness some definitions and properties used in the paper will be given here. For more information see [1, 12–14].

Throughout the paper $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$ the set of non-negative integers. Let R be a nonempty set and let $+$ and $\cdot$ be binary operations on R. An algebraic system $(R, +, \cdot)$ is called a *semiring* if $(R, +, 0)$ is a commutative monoid, $(R, \cdot)$ is a semigroup and multiplication distributes over addition from either side. A semiring $(R, +, \cdot)$ is said to be *commutative* if $\cdot$ is commutative on R. A semiring which is not a ring is called a *proper semiring.*

Zero $0_R \in R$ is called (*multiplicatively*) *absorbing* if $a \cdot 0_R = 0_R \cdot a = 0$ for all $a \in R$. Note that $0_R \in R$ cannot be additively absorbing when R contains more than one element. An element $1_R \in R$ is called *identity* if $a \cdot 1_R = 1_R \cdot a = a$ for all $a \in R$. A subset S of R closed under addition and multiplication is called a *subsemiring* of R.

A *left ideal* of a semiring R is a nonempty set $I \neq R$ which is closed under $+$ and satisfying the following conditions $ra \in I$ for all $a \in I$, $r \in R$. Similarly we can define right ideal and two-sided ideal of a semiring. An ideal I of a semiring R is called *subtractive* (or *k-ideal*) if $a \in I$ and $a + b \in I$ follow $b \in I$.

An element $a \in R$ is called *additively cancellable* if $a + b = a + c$ follows $b = c$ for all $b, c \in R$. Denote by $K^+(R)$ the set of all additively cancellable elements of R. A semiring R is called *additively cancellative* if every element of R is additively cancellable, i. e. $K^+(R) = R$. An element $a \in R$ is called *additively idempotent* if $r + r = r$. Denote by $I^+(R)$ the set of all additively idempotent elements of R. A semiring R is called *additively idempotent* if every element of M is additively idempotent, i. e. $I^+(R) = R$.

Let R be a semiring. A map $\delta\colon R \to R$ is called a *derivation on* R [1] if $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)\, b + a\delta(b)$ for any $a, b \in R$.

A semiring R equipped with a derivation δ is called a *differential semiring* with respect to the derivation δ (or *δ-semiring*), and denoted by (R, δ) [2]. An ideal I of the semiring R is called *differential* if $d(I) \subseteq I$. Any differential semiring has two trivial differential ideals: $\{0_R\}$ and itself.

Throughout the paper we consider commutative semirings.

2. Quasi-prime differential ideals and *dmsp*-semirings. Let A be a subset of R. Denote the smallest differential ideal containing the set A by $[A]$, the smallest radical differential ideal containing A by $\{A\}$, the smallest differential k-ideal containing the set A by $|A|$, and the smallest radical differential k-ideal containing A by $\langle A \rangle$. A non-empty subset S of the semiring R is called a *multiplicatively closed subset* of R if $ab \in S$ for every $a, b \in S$.

Let S be a multiplicatively closed subset of R, $0 \notin S$. If I is a radical differential k-ideal of R maximal among radical differential k-ideals disjoint from S, then I is prime. If I is any radical differential subtractive ideal disjoint from S, then there exists a prime differential k-ideal P containing I which is disjoint from S. If I is a radical differential k-ideal of R, then it is an intersection of all the prime differential k-ideals containing I. [3]

Definition 1. *A differential ideal I of the semiring R is called* quasi-prime *if there exists a multiplicatively closed subset S of R such that I is maximal differential ideal such that $I \cap S = \varnothing$.*

For a subset A of R the set $A_{\#} = \left\{ a \in R \,\middle|\, a^{(n)} \in A \,\text{for all}\, n \in \mathbb{N}_0 \right\}$ is called the *differential* of A.

Examples. Any prime differential ideal of R is quasi-prime. If Q is a maximal differential ideal of R then Q is quasi-prime. In any differential semiring R for any prime ideal P of R the differential ideal $P_{\#}$ is quasi-prime. [3, 4]

Definition 2. *A differential semirings R is called a dmsp-*semiring *(or a Keigher semiring) if for any prime k-ideal P of R, $P_{\#}$ is prime.*

Examples. Every differentially trivial semiring is a *dmsp*-semiring. $\{0\}$ is a *dmsp*-semiring. Any differential semifield is a *dmsp*-semiring. Any Keigher ring is a *dmsp*-semiring.

In a *dmsp*-semiring maximal among differential k-ideals are prime. [3, 4] In a *dmsp*-semiring the radical of an arbitrary differential k-ideal is the intersection of all the prime differential k-ideals containing I. [3]

3. Differentially prime ideals of semirings. We investigate some properties of differentially prime ideals of differential commutative semirings.

Definition 3. *A differential k-ideal P of R is called* differentially prime *if for any $a, b \in R$, $k \in \mathbb{N}_0$, $ab^{(k)} \in P$ follows $a \in P$ or $b \in P$.*

If the ideal I is quasi-prime, then I is a differentially prime ideal of R.

Definition 4. *Let $S \neq \varnothing$ be a subset of R. A subset S is called* d-multiplicatively closed *if for any $a, b \in S$ there exists $n \in \mathbb{N}_0$ such that $ab^{(n)} \in S$.*

If a subset is d-multiplicatively closed, then it is multiplicatively closed.

Proposition 1. *An ideal I of R is differentially prime if and only if $R \setminus I$ is d-multiplicatively closed.*

Proof. Suppose I is a differentially prime ideal of R and there exist $a, b \notin I$ such that $ab^{(n)} \in I$ for all $n \in \mathbb{N}_0$. Then $a \in I$ or $b \in I$, which contradicts to $a, b \in R \setminus I$. Conversely, suppose $R \setminus I$ is d-multiplicatively closed, and for all $a, b \in R$ and all $n \in \mathbb{N}_0$, $ab^{(n)} \in I$, $a, b \notin I$. Then $ab^{(k)} \notin I$ for some $k \in \mathbb{N}_0$, which is a contradiction.

Theorem 1. *For a differential ideal $P \neq R$ of R, the following conditions are equivalent:*

1) *For any $a, b \in R$, $k \in \mathbb{N}_0$, $ab^{(k)} \in P$ follows $a \in P$ or $b \in P$;*
2) *For any $a, b \in R$, $k, l \in \mathbb{N}_0$, $a^{(l)}b^{(k)} \in P$ follows $a \in P$ or $b \in P$;*
3) *For any $a, b \in R$, $[a] \cdot [b] \subseteq P$ follows $a \in P$ or $b \in P$;*
4) *For any differential ideals I and J of R, $IJ \subseteq P$ follows $I \subseteq P$ or $J \subseteq P$.*

Proof. ($1 \Longrightarrow 2$) Suppose $a^{(l)}b^{(k)} \in P$ for any $k, l \in \mathbb{N}_0$. Denote $t = l + k$. For $t = 0$ we have $a^{(0)}b^{(0)} = ab \in P$. Therefore, $\delta(ab) \in P$. For a k-ideal P, $(ab)' = a'b + ab' \in P$, $ab' \in P$ follow $a'b \in P$.

Consider $(ab^{(k)})' = a'b^{(k)} + ab^{(k+1)}$ for all $k \in \mathbb{N}_0$. Similarly, $(ab^{(k)})' \in P$, $ab^{(k+1)} \in P$ follow $a'b^{(k)} \in P$. Then from $(a'b^{(k-1)})' = a''b^{(k-1)} + a'b^{(k)} \in P$, $a'b^{(k)} \in P$ and the subtractiveness of P we obtain $a''b^{(k)} \in P$, etc. ($2 \Longrightarrow 1$) Obvious when $l = 0$.

($2 \Longrightarrow 3$) Conversely, if $[a] \cdot [b] \subseteq P$ then $\sum_{k,l \in \mathbb{N}_0} Ra^{(l)}b^{(k)} \subseteq P$, in particular $a^{(l)}b^{(k)} \in P$. Hence, $a \in P$ or $b \in P$. ($3 \Longrightarrow 2$) It is clear that $[a] = \sum_{l \in \mathbb{N}_0} Ra^{(l)}$, $[b] = \sum_{k \in \mathbb{N}_0} Rb^{(k)}$, and so $[a] \cdot [b] = \sum_{k,l \in \mathbb{N}_0} Ra^{(l)}b^{(k)}$. If $a^{(l)}b^{(k)} \in P$, then $\sum_{k,l \in \mathbb{N}_0} Ra^{(l)}b^{(k)} \subseteq P$. Therefore, $[a] \cdot [b] \subseteq P$, which follows $a \in P$ or $b \in P$.

($3 \Longrightarrow 4$). Suppose $I \nsubseteq P$ and $J \nsubseteq P$. There exists $a \in I$, $a \notin P$, and $b \in J$, $b \notin P$. Clearly, $[a] \cdot [b] \subseteq IJ \subseteq P$. Therefore, $a \in P$ or $b \in P$, which is a contradiction. ($4 \Longrightarrow 3$) is obvious.

Theorem 2. *Let S be d-multiplicatively closed subset of R. If the ideal I is d-maximal in $R \setminus S$, then I is a differentially prime ideal of R.*

Proof. Suppose that there exist $a, b \in R$ and $n \in \mathbb{N}_0$ such that $ab^{(n)} \in N$, $a, b \notin P$. It is clear that $P \subset P + [a]$ and $P \subset P + [b]$. Since P is maximal among the differential ideals not meeting some d-multiplicatively closed subset S, $(P + [a]) \cap S \neq \varnothing$, $(P + [b]) \cap S \neq \varnothing$. Therefore there exist $a, b \in S$ such that $a \in P + [a]$ and $b \in P + [b]$. On the other hand, since S is a d-multiplicatively closed subset, then $a, b \in S$ follows the existence of $n \in \mathbb{N}_0$ such that $ab^{(n)} \in S$. Therefore $b^{(n)} \in (P + [a]) \cap S$. Then $ab^{(n)} \in (P + [a]) \cdot (P + [b]) \subseteq P$. Therefore, $ab^{(n)} \in P \bigcap S \neq \varnothing$, but it contradicts the assumption that $S \bigcap P = \varnothing$. Hence P is a differentially prime ideal.

4. Quasi-prime and differentially prime ideals in Noetherian semirings.

Here we study the interrelationship between differentially prime, quasi-prime and primary ideals of differential Noetherian semirings.

Proposition 2. *Let $a \in R$. There exists $n \in \mathbb{N}_0$ such that $(I : a^n)$ is a differential ideal and $(I : a^n) = (I : a^k)$ for any $k \geq n$.*

Proof. Denote $U = \cup_{l=o}^{\infty}(I : a^l) \subseteq R$. For any $b \in U$ there exists $l \in \mathbb{N}_0$ such that $b \in (I : a^l)$. Then $a^l b \in I$. Since I is a differential ideal of R, then $\delta(a^l b) \in I$. From $\delta(a^l b) = la^{l-1}\delta ab + a^l \delta b \in I$ and subtractiveness of I we have that $a^l \delta(b) \in I$. Thus, $\delta(b) \in (I : a^l)$, which follows $\delta(b) \in U$.

The ideals $(I : a^n)$, $n \in \mathbb{N}_0$, form a chain, therefore there exists $n \in \mathbb{N}_0$ such that $(I : a^n) = (I : a^k)$ for any $k \geq n$.

Theorem 3. *Let R be a Noetherian semiring. If the ideal P of R is differentially prime, then P is a primary ideal of R.*

Proof. Suppose $ab \in P$. For an ideal P we have that $a^n b \in P$, which follows $b \in (P : a^n)$. By Proposition 2, $[b] \subseteq (P : a^n)$. From $a^n b \in P$ we also have that $a^n \in (P : [b])$. Therefore, $[a^n] \subseteq (P : [b])$. Then $[a^n][b] \subseteq P$. By Theorem 1, we have $[a^n] \subseteq P$ or $[b] \subseteq P$. Hence, $a^n \in P$ or $b \in P$, i. e. P is a primary ideal.

Theorem 4. *For every differential ideal I of the Noetherian semiring R the following are equivalent:*
1) *I is a quasi-prime ideal;*
2) *$I = P_{\#}$ for some prime ideal P of R;*
3) *I is a differentially prime ideal.*

Proof. (1) $\Longrightarrow$ (2) Let I be a quasi-prime ideal of R, i. e. maximal among differential ideals disjoint from the multiplicatively closed subset S, and let K be maximal among ordinary ideals disjoint from S and containing I. Then K is prime ideal of R as each d-multiplcatively closed subset is multiplicatively closed. Show that $I = K_{\#}$. Since I is a differential ideal of R, then $I \subseteq K_{\#}$. The converse inclusion implies due to maximality of the differential ideal I among those disjoint from S, because $K_{\#}$ is disjoint from S and it is differential ideal of R.

(2) $\Longrightarrow$ (3) Let $I = P_{\#}$ for some prime ideal of R of M. Then I is maximal among differential ideals disjoint from P. Let $S = R \backslash P$. Assuming that all the derivations are trivial, we see that S is a multiplicatively closed subset of R. Denote by K the intersection of all d-multiplicatively closed subsets of R, which contain S.

Then S is the least d-multiplicatively closed subset of those containing S. Hence I is a differentially prime ideal of R because of 1. It remains to verify that $I = R\backslash K$. Since $R\backslash K$ is disjoint from S, then $R\backslash K \subseteq P$, and due to the fact that $R\backslash K$ is a differential ideal of R, we have the inclusion $R\backslash K \subseteq I$. Given the minimality of the set K, we obtain that the set $R\backslash K$ is a maximal ideal among the differential ideals of I. Hereby $R\backslash K = N$.

(3) $\Longrightarrow$ (1) Let I be some differentially prime ideal of R. Then, by Proposition 1, the set $R\backslash I$ is a d-multiplicatively closed subset of the semiring R. Since I is maximal differential ideal disjoint from $R\backslash I$, then, by definition, it is quasi-prime.

5. Conclusions. This paper is devoted to investigating differentially prime ideals of differential semirings. We continue studying the interrelationship between different types of differential ideals of differential semirings, in particular differentially prime, quasi-prime and primary ideals of differential Noetherian semirings, i. e. differential semirings with the ascending chain conditions for their ideals. The obtained results can be used in the further study of differential ideals.

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Мельник І. О. Про диференціально-первинні ідеали нетерових напівкілець.

Ця стаття присвячена дослідженню поняття `диференціально-первинного ідеалу` в диференціальному комутативному напівкільці (напівкільці разом із заданому на ньому диференціюванням) та його зв'язками з поняттями `квазіпервинного ідеалу` та `примарного ідеалу`. Поняття `диференціювання напівкільця` традиційно визначають як адитивне відображення, яке задовольняє правило Лейбніца, тобто відображення $\delta\colon R \to R$ називають диференціюванням напівкільця R, якщо $\delta\,(a+b) = \delta\,(a) + \delta\,(b)$ і $\delta\,(ab) = \delta\,(a)\,b + a\delta\,(b)$ для будь-яких $a, b \in R$.

Диференціальний ідеал P напівкільця R називають `диференціально-первинним`

ідеалом, якщо для будь-яких $a, b \in R$, $k \in \mathbb{N}_0$, з $ab^{(k)} \in P$ випливає, що $a \in P$ або $b \in P$. Доведено, що ідеал P напівкільця R є диференціально-первинним тоді і тільки тоді, коли для ідеалів I та J напівкільця R з включення $IJ \subseteq P$ випливає, що $I \subseteq P$ або $J \subseteq P$. Квазіпервинний ідеал напівкільця — це диференціальний ідеал, максимальний серед диференціальних ідеалів, що мають порожній перетин з деякою мультиплікативно-замкненою підмножиною даного напівкільця.

У цій статті досліджуються деякі властивості диференціально-первинних ідеалів, зокрема таких ідеалів в диференціальних нетерових напівкільцях.

Стаття складається з двох основних частин. У першій частині встановлено деякі властивості диференціально-первинних ідеалів та подано приклади таких ідеалів. У другій частині статті автор досліджує зв'язки, що існують між поняттями квазіпервинний, примарний ідеал та диференціально-первинний ідеал в нетерових диференціальних напівкільцях. Встановлено, що в диференціальному нетеровому напівкільці R диференціальний ідеал I напівкільця R є диференціально-первинним ідеалом тоді і тільки тоді, коли I є квазіпервинний ідеал.

Ключові слова: диференціювання напівкільця, диференціальне напівкільце, диференціальний ідеал напівкільця, диференціально-первинний ідеал, квазіпервинний ідеал, примарний ідеал, нетерове напівкільце.

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