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V. M. Bondarenko[1], M. V. Stoika[2], M. V. Styopochkina[3]

[1] Institute of Mathematics of NAS of Ukraine,
Leading researcher of the department of algebra and topology,
Doctor of physical and mathematical sciences
vitalij.bond@gmail.com
ORCID: https://orcid.org/0000-0002-5064-9452

[2] Ferenc Rakoczi II. Transcarpathian Hungarian Institute,
Associate professor of the department of mathematics and informatics,
Candidate of physical and mathematical sciences
stoyka_m@yahoo.com
ORCID: https://orcid.org/0000-0002-0840-1496

[3] Polissia National University,
Associate professor of the department of higher and applied mathematics,
Candidate of physical and mathematical sciences
stmar@ukr.net
ORCID: https://orcid.org/0000-0002-7270-9874

THE COEFFICIENTS OF TRANSITIVITY OF THE POSETS OF *MM*-TYPE BEING THE HIGHEST SYPERCRITICAL POSET

The representations of partially ordered sets (abbreviated as posets), introduced by L. A. Nazarova and A. V. Roiter (in matrix form) in 1972, play an important role in the modern representation theory. In his first paper on this topic M. M. Kleiner proved that a poset S is of finite representation type (i.e. has a finite number, up to equivalence, of indecomposable representations) if and only if it does not contain subposets of the form $\mathcal{K}_1 = (1,1,1,1), \mathcal{K}_2 = (2,2,2), \mathcal{K}_3 = (1,3,3), \mathcal{K}_4 = (1,2,5)$ and $\mathcal{K}_5 = (\mathrm{N},4)$. Specified posets are called critical posets relative to the finiteness of the type. They are also called the Kleiner's (critical) posets. In 1974 Yu. A. Drozd proved that a poset S has finite representation type if and only if its Tits quadratic form

$$q_S(z) =: z_0^2 + \sum_{i\in S} z_i^2 + \sum_{i<j,i,j\in S} z_i z_j - z_0 \sum_{i\in S} z_i$$

is weakly positive (i.e., positive on the set of non-negative vectors). Consequently, the Kleiner's posets are critical relative to weak positivity of the Tits quadratic form, and there are no (up to isomorphism) other such posets. In 2005 the authors proved that a poset is critical relative to the positivity of the Tits quadratic form if and only if it is minimax isomorphic to a Kleiner's poset.

A similar situation takes place for posets of tame representation type. In 1975 L. A. Nazarova proved that a poset S is tame if and only if it does not contain subsets of the form $\mathcal{N}_1 = (1,1,1,1,1)$, $\mathcal{N}_2 = (1,1,1,2)$, $\mathcal{N}_3 = (2,2,3)$, $\mathcal{N}_4 = (1,3,4)$, $\mathcal{N}_5 = (1,2,6)$ and $\mathcal{N}_6 = (\mathrm{N},5)$. She called these posets supercritical; they are also critical relative to weak non-negativity of the Tits quadratic. In 2008 the authors proved that a poset is critical relative to non-negativity of the Tits quadratic form if and only if it is minimax isomorphic to a supercritical poset.

In this paper we study the combinatorial properties of posets, minimax isomorphic to the supercritical poset of greatest height, i.e. $(1,2,6)$. The importance of studying minimax isomorphic posets is determined by the fact that their Tits quadratic forms are $\mathbb{Z}$-equivalent, and minimax isomorphism itself is a fairly general constructively defined $\mathbb{Z}$-equivalence of the Tits quadratic forms for posets.

Keywords: representation, critical and supercritical poset, Tits quadratic form, finite and tame representation type, positivity and weak positivity, non-negativity and weak non-negativity, minimax isomorphism, coefficient of transitivity.

1. Introduction. The representations of partially ordered sets (abbreviated as posets), introduced by L. A. Nazarova and A. V. Roiter (in matrix form) in 1972 [1], play an important role in the modern representation theory. In his first paper on this topic M. M. Kleiner [2] proved that a posets S is of finite representation type (i.e. has a finite number, up to equivalence, of indecomposable representations) if and only if it does not contain subposets of the form $\mathcal{K}_1 = (1,1,1,1), \mathcal{K}_2 = (2,2,2), \mathcal{K}_3 = (1,3,3), \mathcal{K}_4 = (1,2,5)$ and $\mathcal{K}_5 = (\mathrm{N}, 4)$. Specified posets are called *critical posets* relative to the finiteness of the type (i,e. they are minimal posets with an infinite number of indecomposable representations, up to equivalence). They are also called the *Kleiner's (critical) posets*. On the other hand, Yu. A. Drozd [3] proved that a poset has finite representation type if and only if its Tits quadratic form is weakly positive (i.e., positive on the set of nonnegative vectors). From these two statements it follows that the critical posets are also critical relatively to the weak positivity of the Tits quadratic form.

In 2005 the authors proved that a poset is critical relatively to the positivity of the Tits quadratic form if and only if it is minimax isomorphic to a Kleiner's poset [4] (such isomorphism was introduced by the first author in [5]); in this paper all such posets, which were named by the authors as P-critical, were fully described.

A similar situation takes place in the case of tame posets. A poset S is tame if and only if it does not contain subsets of the form $\mathcal{N}_1 = (1,1,1,1,1)$, $\mathcal{N}_2 = (1,1,1,2)$, $\mathcal{N}_3 = (2,2,3)$, $\mathcal{N}_4 = (1,3,4)$, $\mathcal{N}_5 = (1,2,6)$, $\mathcal{N}_6 = (\mathrm{N}, 5)$ [6], and this is equivalent to the weak non-negativity of the Tits quadratic form of S; these posets are called *supercritical*. In [7] the authors proved that a poset is critical relatively to the non-negativity of the Tits quadratic form if and only if it is minimax isomorphic to a supercritical poset. In [8] all such posets, which were named by the authors as NP-critical, were fully described.

In addition to the above three papers of the authors, posets with positive and non-negative Tits quadratic form. were also studied in many of their other papers (see e.g. [9] – [17]).

In this paper, continuing to study the combinatorial properties of different classes of posets (see e.g. [18] – [22]), we consider posets that are minimax isomorphic to the supercritical poset of greatest height.

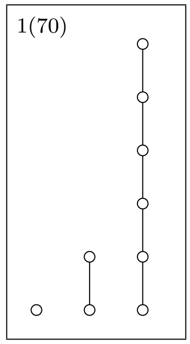
2. The list of posets of MM-type $(1,2,6)$. Throught the paper, all posets will be supposed finite.

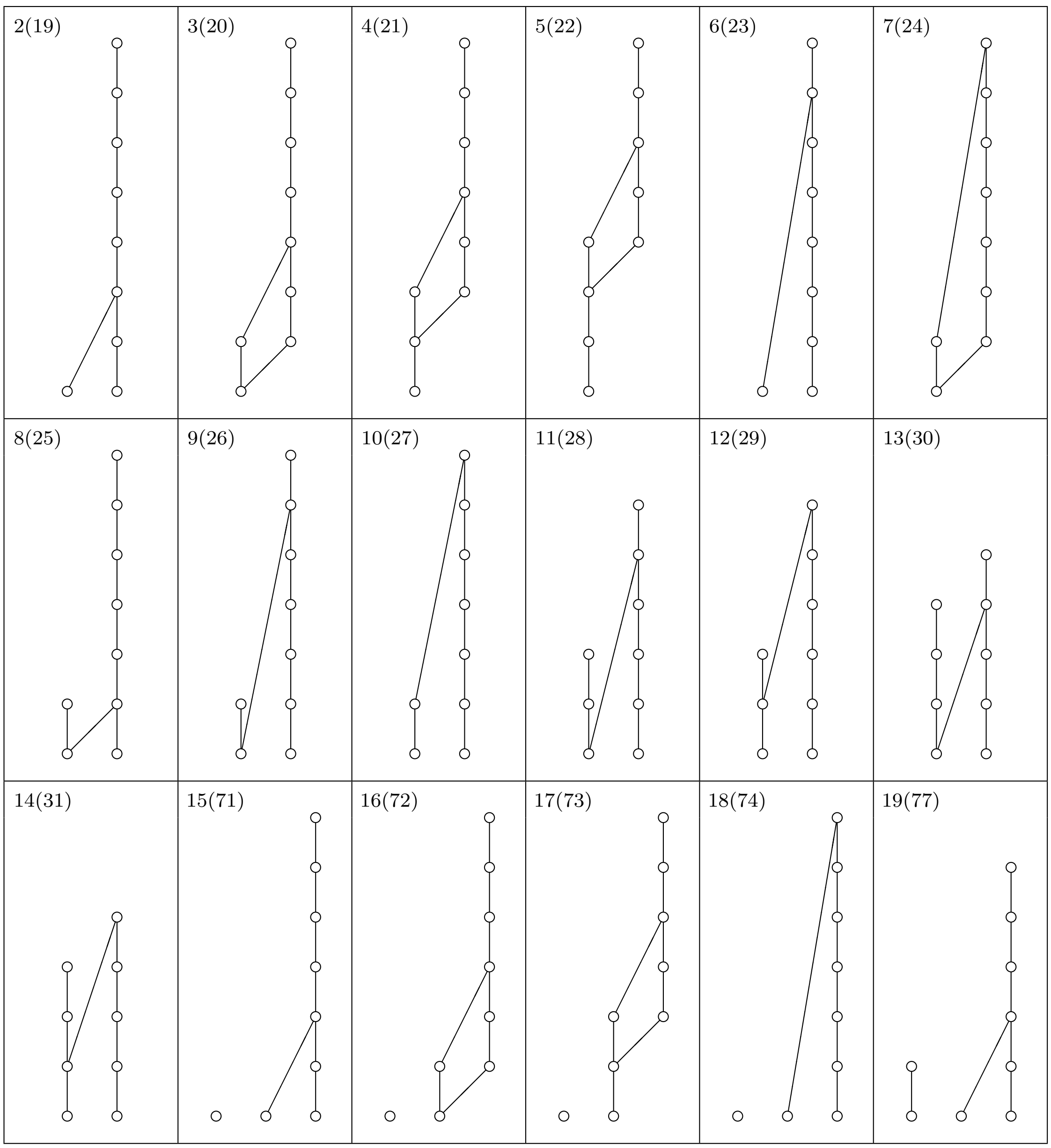
Let S be a poset. For a minimal (resp. maximal) element a of S, denote by $T = S_a^{\uparrow}$ (respect. $T = S_a^{\downarrow}$) the following poset: $T = S$ as usual sets, $T \setminus a = S \setminus a$ as posets, the element a is maximal (resp. minimal) in T, and a is comparable with x in T if and only if they are incomparable in S. Two posets S and T are called (min, max)-*equivalent* if there are posets $S_1, \ldots, S_p$ $(p \geq 0)$ such that, if we put $S = S_0$ and $T = S_{p+1}$, then, for every $i = 0, 1, \ldots, p$, either $S_{i+1} = (S_i)_{x_i}^{\uparrow}$ or $S_{i+1} = (S_i)_{y_i}^{\downarrow}$ [5]. Obviously, any poset is (min, max)-equivalent to itself. Since some time we also use the term *minimax equivalence*.

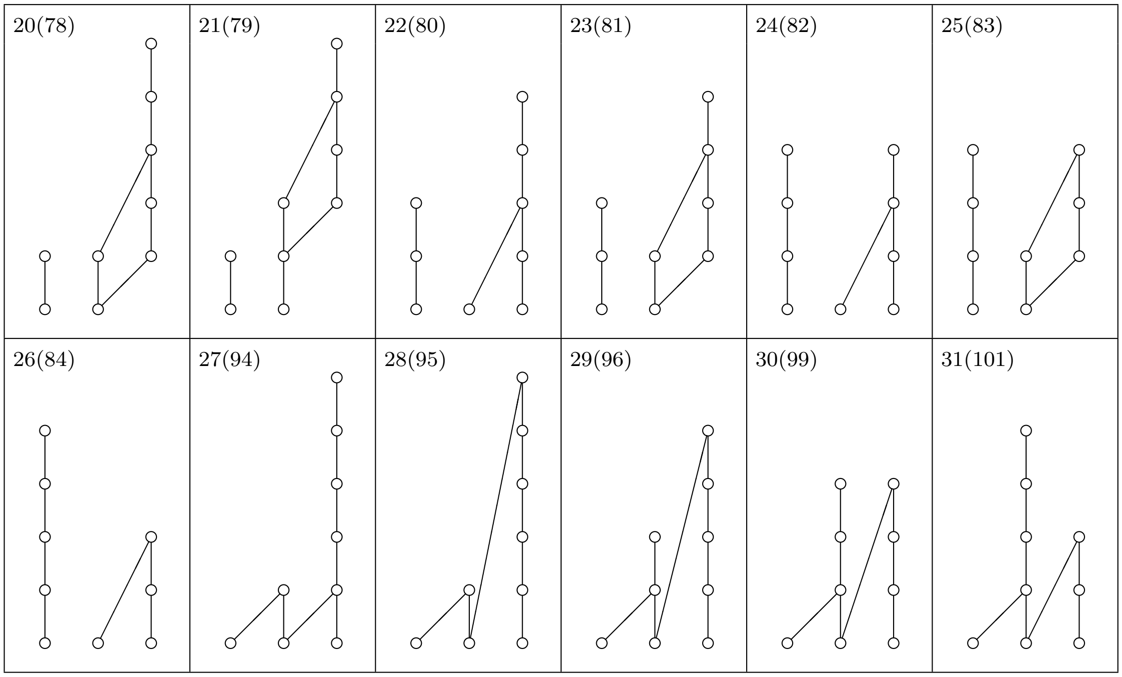
The notion of minimax equivalence can be naturally continued to the notion of *minimax isomorphism*: posets S and S' are minimax isomorphic if there exists a poset T, which is minimax equivalent to S and isomorphic to S'.

When P is a fix poset, we call that a poset S is of *MM-type* P if S is minimax isomorphic to P [19].

From the results of [8] it follows that the following table contains all posets (up to isomorphism and duality) of MM-type $(1, 2, 6)$, which is the highest supercritical poset (in the sense of its longest chain):







3. Coefficients of transitivity. Main results. Let S be a poset and $S^2_< := \{(x,y)\,|\,x,y \in S, x < y\}$. If $(x,y) \in S^2_<$ and there is no z satisfying $x < z < y$, then we say that x and y are *neighboring*. Put $n_w = n_w(S) := |S^2_<|$ and denote by $n_e = n_e(S)$ the number of pairs of neighboring elements. The ratio $k_t = k_t(S)$ of the numbers $n_w - n_e$ and n_w are called the *coefficient of transitiveness of* S; if $n_w = 0$ (then $n_e = 0$), we assume $k_t = 0$ (this notion was introduced in [18]).

In this part of the paper we calculate k_t for all posets of MM-type $\mathcal{N}_5 = (1,2,6) = \{1,2,\cdots,9\,|\,2 \prec 3, 4 \prec 5 \prec 6 \prec 7 \prec 8 \prec 9\}$, which is the highest supercritical one.

Theorem 1. *The following holds for posets* 1 – 31 *of* MM*-type* $\mathcal{N}_5$*:*

N	n_e	n_w	k_t	N	n_e	n_w	k_t	N	n_e	n_w	k_t
1	6	16	0,625	11	8	20	0,6	22	7	16	0,5625
2	8	34	0,76471	12	8	20	0,6	23	8	16	0,5
3	9	34	0,73529	13	8	18	0,55556	24	7	14	0,5
4	9	34	0,73529	14	8	18	0,55556	25	8	14	0,42857
5	9	34	0,73529	15	7	26	0,73077	26	7	14	0,5
6	8	30	0,73333	16	8	26	0,69231	27	8	22	0,63636
7	9	30	0,7	17	8	26	0,69231	28	8	18	0,55556
8	8	28	0,71429	18	7	22	0,68182	29	8	16	0,5
9	8	24	0,66667	19	7	20	0,65	30	8	16	0,5
10	8	24	0,66667	20	8	20	0,6	31	8	18	0,55556
				21	8	20	0,6				

The transitivity coefficients are written out with an accuracy of five decimal places. The value is exact if and only if the number of decimal places is less than five, and two values equal to exactly five digits are equal at all.

The proof is carried out by direct calculations.

Recall that the greatest length among the lengths of all linear ordered subsets of a poset S is called its *height* An element of a poset is called *nodal*, if it is comparable with all the others elements. A subposet X of T is said to be *dense* if there is not $x_1, x_2 \in X$, $y \in T \setminus X$ such that $x_1 < y < x_2$.

Note that a poset of MM-type $\mathcal{N}_5$ can have at most six nodal elements.

Corollary 1. *The coefficient $k_t(S)$ of a poset S is the largest among all the posets of MM-type $\mathcal{N}_5$ if and only if S contains a dense subposet with six nodal elements.*

Corollary 2. *The coefficient $k_t(S)$ of a poset S is the smallest among all the posets of MM-type $\mathcal{N}_5$ if and only if S is a self-dual poset of height four.*

4. Conclusions. In this paper we investigate combinatorial aspects of supercritical posets which arise in the study of tame posets. Namely, we indicate, up to isomorphism and duality, all the posets that are minimax isomorphic to the supercritical posets $(1, 2, 6)$ and describe for them the coefficients of transitivity.

The importance of studying minimax isomorphic posets is determined by the fact that their Tits quadratic forms are $\mathbb{Z}$-equivalent. Using this fact the authors, in particular, proved (earlier) that a poset is critical relative to the positivity of the Tits quadratic form if and only if it is minimax isomorphic to a critical Kleiner's poset, and a poset is critical relatively to the non-negativity of the Tits quadratic form if and only if it is minimax isomorphic to a supercritical poset. They also described all the posets that are minimax isomorphic to any critical or supercritical poset, classified all the posets with positive Tits quadratic form and solved a number of other classification problems,

The obtained results can be used in the study of combinatorial aspects of other classes of posets.

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Бондаренко В. М., Стойка М. В., Стьопочкіна М. В. Коефіцієнти транзитивності частково впорядкованих множин найвищого суперкритичного MM-типу.

Зображення частково впорядкованих (скорочено ч. в.) множин, які введені Л. А. Назаровою і А. В. Ройтером (в матричній формі) в 1972 р., відіграють важливу роль в сучасній теорії зображень. У своїй першій праці за цією тематикою М. М. Клейнер довів, що ч. в. множина S має скінченний зображувальний тип (тобто має скінченне число нерозкладних зображень, з точністю до еквівалентності) тоді і лише тоді, коли вона не містить ч. в. підмножин вигляду $K_1 = (1,1,1,1), K_2 = (2,2,2), K_3 = (1,3,3), K_4 = (1,2,5)$ і $K_5 = (\mathrm{N},4)$. Вказані ч. в. множини називаються критичними ч. в. множин щодо скінченності типу (тобто вони є мінімальними ч. в. множинами з нескінченним числом нерозкладних зображень, з точністю до еквівалентності). Їх також називають (критичними) ч. в. множинами Клейнера. У 1974 р. Ю. А. Дрозд довів, що ч. в. множина S має скінченний зображувальний тип тоді і лише тоді, коли її квадратична форма Тітса

$$q_S(z) =: z_0^2 + \sum_{i\ inS} z_i^2 + \sum_{i<j,i,j\ inS} z_i z_j - z_0 \sum_{i\ inS} z_i$$

є слабко додатною (тобто додатною на множині невід'ємних векторів). Таким чином, ч. в. множини Клейнера є критичними щодо слабкої додатності квадратичної форми Тітса, і інших таких ч. в. множин немає (з точністю до ізоморфізму). У 2005 р. автори

довели що ч. в. множина є критичною щодо додатності квадратичної форми Титса тоді і лише тоді, коли вона мінімаксно ізоморфна деякій ч. в. множині Клейнера.

Подібну ситуацію маємо з ч. в. множинами ручного зображувального типу. У 1975 р. Л. А. Назарова довела, що ч. в. множина S є ручною тоді і лише тоді, коли вона не містить ч. в. підмножин вигляду $N_1 = (1,1,1,1,1)$, $N_2 = (1,1,1,2)$, $N_3 = (2,2,3)$, $N_4 = (1,3,4)$, $N_5 = (1,2,6)$ і $(\mathrm{N},5)$. Вона назвала ці ч. в. множини суперкритичними; вони є також критичними щодо слабкої невід'ємності квадратичної форми Тітса. У 2009 році автори довели, що ч. в. множина є критичною щодо невід'ємності квадратичної форми Тітса тоді і лише тоді, коли вона мінімаксно ізоморфна деякій суперкритичній ч. в. множині.

У цій статті вивчаються комбінаторні властивості ч. в. множин, мінімаксно ізоморфних суперкритичній ч. в. множині найбільшої висоти, тобто $(1,2,6)$. Важливість вивчення мінімаксно ізоморфних ч. в. множин визначається тим фактом, що їх квадратичні форми Тітса $\mathbb{Z}$-еквівалентні, а сам мінімаксний ізоморфізм є досить загальною конструктивно визначеною $\mathbb{Z}$-еквівалентністю для квадратичних форм Тітса ч. в. множин.

Ключові слова: зображення, критична і суперкритична ч. в. множина, квадратична форма Тітса, скінченний і ручний зображувальний тип, додатність і слабка додатність, негативність і слабка негативність, мінімаксний ізоморфізм, коефіцієнт транзитивності.

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