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ON A CRITERION OF THE FINITENESS OF THE REPRESENTATION TYPE FOR FAMILIES OF THE CATEGORIES OF INJECTIVE REPRESENTATIONS

The representations of posets (partially ordered sets), introduced by L. A. Nazarova and A. V. Roiter in 1972, play an important role in the modern representation theory and its applications. M. M. Kleiner obtained a description of posets of finite representation type in terms of critical posets (the minimal ones of infinite representation type) and Yu. A. Drozd proved that a poset S (not containing an element designated as 0) is of finite representation type if and only if its Tits quadratic form

$$q_S(z) =: z_0^2 + \sum_{i\in S} z_i^2 + \sum_{i<j,i,j\in S} z_i z_j - z_0 \sum_{i\in S} z_i$$

is weakly positive, i.e. positive on the set of non-negative vectors (in 1972 and 1974, respectively). In this paper we consider a situation (which deals with infinite posets), when the main role is played not by weakly positivity but by positivity of the Tits quadratic form. The situation relates to the study of the categories of representations of a special form, and in this case we use established by the first author a connection between the Tits quadratic forms for partially ordered sets and commutative quivers.

Keywords: injective representation, critical poset, Tits quadratic form for posets, Tits quadratic form for commutative quivers, finite representation type, positivity and weak positivity.

1. Introduction. The representations of partially ordered sets (abbreviated as posets), introduced by L. A. Nazarova and A. V. Roiter (in matrix form) in 1972 [1], play an important role in the modern representation theory. In his first paper on this topic M. M. Kleiner [2] proved that a posets S is of finite representation type (i.e. has, up to equivalence, a finite number of indecomposable representations) if and only if it does not contain subposets of the form $\mathcal{K}_1 = (1,1,1,1), \mathcal{K}_2 = (2,2,2), \mathcal{K}_3 = (1,3,3), \mathcal{K}_4 = (1,2,5)$ and $\mathcal{K}_5 = (\mathrm{N},4)$. Specified posets are called the *critical posets* relative to the finiteness of the type (i.e. they exhaust all the minimal posets with an infinite number of indecomposable representations, up to equivalence). On the other hand, in 1974 Yu. A. Drozd [3] proved that a poset has finite representation type if and only if the Tits quadratic form

$$q_S(z) =: z_0^2 + \sum_{i\in S} z_i^2 + \sum_{i<j,i,j\in S} z_i z_j - z_0 \sum_{i\in S} z_i$$

is weakly positive (i.e., positive on the set of non-negative vectors). From these two statements it follows that the critical posets are also critical relatively to the weak positivity of the above quadratic form.

We single out the main further works of Kyiv mathematicians on this topic [4] – [14], which is related to the above indicated papers (limited to the period of 30 years and without claiming the completeness of this list).

In 2005 the authors [15] proved that a poset is critical relatively to the positivity of the Tits quadratic form if and only if it is minimax isomorphic (in the sence of [16]) to a Kleiner's poset; in [15] all such posets and also posets with their quadratic forms to be positive were fully described.

In this paper, which is naturally considered as a continuation of the papers [17] and [18], we study a situation (dealing with infinite posets), when the main role is played not by weakly positivity but by positivity of the Tits quadratic form. The situation relates to the study of the categories of representations of a special form, and in this case we use established by the first author a connection between the Tits quadratic forms for posets and commutative quivers.

2. Representations of posets. Throughout the paper, k denotes a field and all k-vector spaces are finite-dimensional. The category of k-vector spaces is denoted by $\mathrm{mod}\,k$. Linear mappings and morphisms of categories multiply from left to right. For formal reasons, we always assume that a poset does not contain an element designated as 0 or $+\infty$.

Recall the well-known definitions about representations of posets in terms of vector spaces graded by posets (see [13]).

Let A be a finite poset. An *A-graded vector space over k* is by definition the direct sum $U = \bigoplus_{a\in A} U_a$ of k-vector spaces U_a. A linear map $\varphi : U \to U'$ between A-graded vector spaces U and U' is called an *A-map* if $\varphi_{bc} = 0$ for each $b, c \in A$ not satisfying $b \le c$, where φ_{xy} denotes the linear map of U_x into U'_y induced by the map φ.

A *representation of a poset A over k* is a triple $W = (V, U, \gamma)$ formed by a k-vector space V, an A-graded space U and a linear map $\gamma : V \to U$; a morphism of representations $W \to W'$ is a pair (μ, ν), formed by a linear map $\mu : V \to V'$ and an A-map $\nu : U \to U'$, such that $\gamma\nu = \mu\gamma'$. The category of representations of A will be denoted by $Rep_k A$.

Injective and projective representations of posets are defined in a standard way. In this paper, we are interested in injective representations.

For representations X and Y of a poset A, we write $0 \Rightarrow X \overset{\alpha}{\to} Y$ if all maps $\varphi_{xx}, x \in A$, are injective. A representation X of a poset A is said to be *injective* if any diagram

$$\begin{array}{ccccc} 0 & \Rightarrow & R' & \to & R \\ & & \downarrow & & \\ & & X & & \end{array}$$

can be embedded in a commutative diagram

$$\begin{array}{ccccc} 0 & \Rightarrow & R' & \to & R \\ & & \downarrow & \swarrow & \\ & & X & & \end{array} .$$

The full subcategory of Rep_kA consisting of all injective objects will be denoted by Inj_kA. The poset A is said to be of *inj-finite representation type over* k if the category $Funct(Inj_kA, \mathrm{mod}\, k)$ of functors from the category Inj_kA to the category $\mathrm{mod}\, k$, which is called (according to the general definition for categories) the category of representations of Inj_kA, is of finite type, i.e. has, up to isomorphism, a finite number of indecomposable objects.

3. Main result. Let S be an infinite poset and $\mathbb{Z}$ denotes the integer numbers. Denote by $\mathbb{Z}_0^{S\cup 0}$ the subset of the cartesian product $\mathbb{Z}^{S\cup 0} = \{z = (z_i) \,|\, i \in S \cup 0\}$ consisting of all vectors $z = (z_i)$ with finite number of nonzero coordinates. We call *the quadratic Tits form of* S (by analogy with the case of a finite poset) the form $q_S : \mathbb{Z}_0^{S\cup 0} \to \mathbb{Z}$ defined by the equality

$$q_S(z) = z_0^2 + \sum_{i\in S} z_i^2 + \sum_{i<j, i,j\in S} z_i z_j - z_0 \sum_{i\in S} z_i.$$

This form is called *positive* if it take positive values for all nonzero $z \in \mathbb{Z}_0^{S\cup 0}$.

We formulate now the main result of this paper.

Theorem 1. *Let S be an unlimited poset, i.e. it has no both the minimal and maximal elements, and k be a field. Then the following conditions are equivalent:*

(I) every finite subposet of S is of inj-finite representation type over k;

(II) the Tits quadratic form of S is positive.

4. Representations of commutative quivers and their connections with injective representations of posets. Let $Q = (Q_0, Q_1)$ be a finite quiver with the set. of vertices Q_0 and the set of arrows Q_1.

A representation $\overline{U}$ of the quiver $Q = (Q_0, Q_1)$ over a field k consists of vector spaces $U_i \in \mathrm{mod}\, k, i \in Q_0$, and linear mappings $\gamma_\alpha : U_x \to U_y$, where $\alpha : x \to y$ runs through Q_1. Morphism φ from $\overline{U}$ to $\overline{U'}$ consists of linear mappings $\varphi_x : U_x \to$ $\to U'_x, x \in Q_0$, such that for each arrow $\alpha : x \to y$ the diagram

$$\begin{array}{ccc} U_x & \xrightarrow{\gamma_\alpha} & U_y \\ {\scriptstyle\varphi_x}\downarrow & & \downarrow{\scriptstyle\varphi_y} \\ U'_x & \xrightarrow{\gamma'_\alpha} & U'_y \end{array}$$

is commutative. The category representations over k of the quiver Q is denoted by Rep_kQ. The quiver Q is said to be of finite representation type over k if the category Rep_kQ is of finite type.

A quiver $Q = (Q_0, Q_1)$ is called *commutative* if it has no multiple arrows and oriented cycles, and any two path with the same starting and terminating vertices are equal (it is assumed that there are no other relations on the paths).

The Tits quadratic form $br_Q(z)$ of a commutative quiver $Q = (Q_0, Q_1)$, the study of which was initiated by S. Brenner [19], differs from the Tits quadratic form $q_Q(z)$ of $Q = (Q_0, Q_1)$ as an usual quiver [20] by the presence of an additional term, which depends on the number of independent relations (in both cases $z = (z_1, \dots, z_n)$, where $n = |Q_0|$). Consider this situation more precisely; we shall look at it based on [19]–[22].

Denote by kQ the k-algebra of paths, whose basis is all paths on the quiver Q, by J the ideal in it generated by all arrows of Q, and by I the ideal generated by all elements $f - g \in kQ$, where f and g are paths with the same starting and terminating vertices. Then by definition the quadratic Tits form $br_Q : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ of the commutative quiver Q is defined by the equality

$$br_Q(z) =: q_Q(z) + \sum_{i,j \in S} r_{ij} z_i z_j = \sum_{i \in Q_0} z_i^2 - \sum_{(i \to j) \in Q_1} z_i z_j, + \sum_{(i,j \in S} r_{ij} z_i z_j$$

with $r_{ij} = dim_k e_i(I/IJ + JI)e_j$, where e_s denotes the primitive idempotent of kQ corresponding to the trivial path $s \to s$.

Theorem 2. *Let Q be a (finite) commutative quiver and k be an arbitrary field. Then the following condition are equivalent:*

(a) Q is of finite representation type;

(b) the Tits quadratic form $br_Q(z)$ is weakly positive.

Note that a similar theorem holds for an usual quiver and the Tits quadratic form $q_Q(z)$ (Gabriel's theorem [20]) but in this case the weak positivity is equivalent to the positivity. As indicated in [17], Theorem 2 follows from the resulfs of [23]. This is also mentioned in [21], but the corresponding calculations are not given in [23].

Let now S be a finite poset. We associate to S the quiver $\overrightarrow{S} = (\overrightarrow{S}_0, \overrightarrow{S}_1)$ with the set of vertices $\overrightarrow{S}_0$ consisting of the elements of S and the set of arrows

$$\overrightarrow{S}_1 = \{i \to j \,|\, i < j,\ i \text{ and } j \text{ are adjacent}\}$$

(elements i and $j > i$ is called *adjacent* if there is not an element s with $j > s > i$). We always shall consider the quiver $\overrightarrow{S} = (\overrightarrow{S}_0, \overrightarrow{S}_1)$ as a commutative one.

Theorem 3 ([17])**.** *Let S be a finite poset and k be a field. Denote by S^+ the poset $S \cup +\infty$ with $x < +\infty$ for any $x \in S$. Then the following conditions are equivalent:*

(1) the poset S is of inj-finite representation type;

(2) the commutative quiver $\overrightarrow{S^+}$ is of finite representation type.

5. Proof of Theorem 1. Theorem 1 follows from the above theorems and the following one:

Theorem 4 ([24])**.** *Let S be an unlimited poset. Then the following conditions are equivalent:*

(A) the Tits quadratic form $q_S(z)$ is positive for any finite subposet $P \subset S$;

(B) the Tits quadratic form $br_{\overrightarrow{S}}(z)$ is positive for any finite subposet $P \subset S$;

(C) the Tits quadratic form $br_{\overrightarrow{S}}(z)$ is weakly positive for any finite subposet $P \subset S$.

Namely, the implication $(I) \Rightarrow (II)$ follows from the implications $(1) \Rightarrow (2)$ (Theorem 3), $(a) \Rightarrow (b)$ (Theorem 2) and $(B) \Rightarrow (A)$ (Theorem 4); the implication $(II) \Rightarrow (I)$ follows from the implications $(A) \Rightarrow (C)$ (Theorem 4), $(b) \Rightarrow (a)$ (Theorem 2) and $(2) \Rightarrow (1)$ (Theorem 3).

6. Conclusions. In this paper we study representations of posets and consider a dealing with infinite posets situation, when the main role is played not by weakly positivity (as in the cases of finite posets) but by positivity of the Tits quadratic form. The situation relates to the investigation of subcategories of injective objects in the categories of representations of posets.

We prove that every finite subposet of an unlimited poset is of inj-finite representation type over a field k (i.e. the category of injective representations has, up to isomorphism, a finite number of indecomposable objects) if and only if the Tits quadratic form of S is positive. In this proof, an impotent role is played by commutative quivers and their Tits quadratic form.

The obtained results can be used in the study of similar problems.

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Бондаренко В. М., Стьопочкіна М. В. Про критерій скінченності зображувального типу для сімейств категорій ін'єктивних зображень.

Зображення ч. в. множин (частково впорядкованих множин), введені Л. А. Назаровою і А. В. Ройтером у 1972 р., відіграють важливу роль у сучасній теорії зображень та її застосуваннях. М. М. Клейнер отримав опис ч. в. множин скінченного зображувального типу в термінах критичних ч. в . множин (мінімальних ч. в. множин нескінченного зображувального типу), а Ю. А. Дрозд довів, що ч. в. множина S (яка не містить елемента, позначеного як 0) має скінченний зображувальний тип тоді і тільки тоді, коли її квадратична форма Тітса

$$q_S(z) =: z_0^2 + \sum_{i\in S} z_i^2 + \sum_{i<j, i,j\in S} z_i z_j - z_0 \sum_{i\in S} z_i$$

є слабко додатною, тобто додатною на множині невід'ємних векторів (у 1972 та 1974 роках відповідно). У цій статті ми розглядаємо ситуацію (що стосується нескінченних ч. в. множин), коли головну роль відіграє не слабка додатність, а додатність квадратичної форми Тітса. Ситуація стосується дослідження категорій зображень спеціального вигляду, і в цьому випадку ми використовуємо встановлений першим автором зв'язок між квадратичними формами Тітса для частково впорядкованих множин і комутативних сагайдаків.

Ключові слова: ін'єктивне зображення, критична ч. в. множина, квадратична форма Тітса для ч. в. множин, квадратична форма Тітса для комутативних сагайдаків, скінченний зображкувальний тип, додатність і слабка додатність.

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