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SKEW COUNTERPARTS OF THE GENERALIZED DOUBLE LOMAX DISTRIBUTION: PROPERTIES AND APPLICATIONS

Two extensions of the generalized double Lomax distribution are introduced — the spliced-scale generalized double Lomax distribution and the exponentiated generalized double Lomax distribution. Their properties are studied. Usefulness of the new distributions is shown by fitting them to stock returns datasets.

Keywords: generalized double Lomax distribution, exponentiated distribution, maximum likelihood, moment, entropy.

1. Introduction. The generalized double Lomax (GDL) distribution was introduced in [1]. This distribution is a symmetric analog of the Lomax distribution. Its pdf is

$$p(x) = \frac{v}{2s}\left(1 + \frac{|x-m|}{s}\right)^{-v-1}, \quad x \in \mathbb{R},$$

where m and $v > 0$, $s > 0$ are the distribution parameters.

Notation GDL(v, m, s) will be used afterwards for the GDL distribution with the parameters v, m, s.

The generalized double Lomax distribution was successfully fitted to datasets of daily returns for several stock indexes and equities in [1].

Although the GDL distribution is capable of modeling real data, it is interesting to obtain extensions of this distribution which are more flexible. Asymmetric counterparts of the GDL distribution would be especially useful (such skew generalizations, in particular, would be more suitable for modeling of stock returns distributions).

There are many ways of creating new distribution families (many of which can be used, in particular, for skewing a symmetric distribution) — see, for instance, [2] and [3], [4], [5]. We will use exponentiation (see [6]) and "scale splicing" (proposed in [7]) for obtaining new families.

Two skew extensions of the generalized double Lomax distribution will be introduced — the exponentiated generalized double Lomax (EGDL) distribution and the spliced-scale generalized double Lomax (SpScGDL) distribution. Properties of these distributions will be analyzed. Usefulness of these counterparts will be demonstrated by providing financial datasets which can be modeled adequately by the exponentiated generalized double Lomax distribution and the spliced-scale generalized double Lomax distribution. Goodness-of-fit statistics for EGDL and SpScDL distributions will be compared to those of other distribution families.

2. Spliced-scale generalized double Lomax distribution. We will introduce the spliced-scale generalized double Lomax distribution by skewing the generalized double Lomax distribution according to the approach of Fernandez and Steel (see [7]). Namely, suppose that a symmetric unimodal distribution has the pdf $p(x)$. Fernandez and Steel define the pdf of the skewed distribution as

$$p_\gamma(x) = \frac{2}{\gamma + 1/\gamma}\Big(p(x/\gamma) I_{[0;\infty)}(x) + p(\gamma x) I_{(-\infty;0)}(x)\Big),$$

where $\gamma \in (0;\infty)$ controls the skewness of this distribution.

Definition 1. *The spliced-scale generalized double Lomax distribution with the parameters τ, v, m and s $(\tau, v, s > 0)$ (or* SpScGDL(τ, v, m, s) *distribution) is defined as the distribution with the pdf*

$$p_{\mathrm{SpSc}}(x; \tau, v, m, s) = \begin{cases} c_\tau \dfrac{v}{2s}\left(1 - \dfrac{x-m}{s\tau^2}\right)^{-v-1}, & \text{if } x < m; \\ c_\tau \dfrac{v}{2s}\left(1 + \tau^2 \dfrac{x-m}{s}\right)^{-v-1}, & \text{if } x \ge m, \end{cases}$$

where

$$c_\tau = \frac{2}{\tau^2 + 1/\tau^2}. \tag{1}$$

We will write $p_{\mathrm{SpSc}}(x)$ instead of $p_{\mathrm{SpSc}}(x; \tau, v, m, s)$ if this causes no confusion.

Remark 1. *The case $\tau = 1$ corresponds to the* GDL(v, m, s) *distribution.*

Figures 1 and 2 show the densities of several spliced-scale generalized double Lomax distributions.

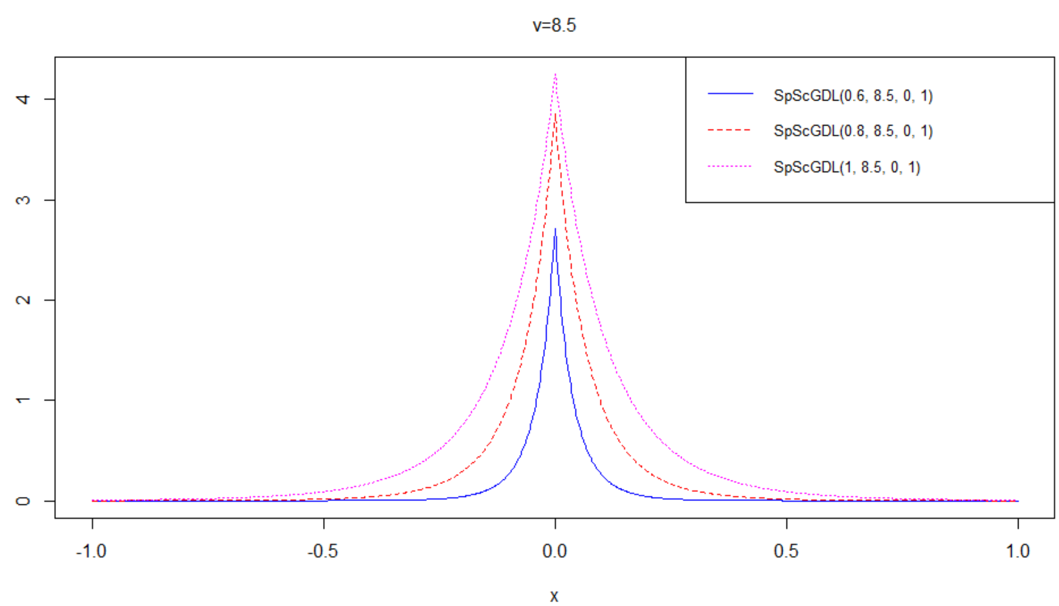

Figure 1. The densities of SpScGDL distributions with $\tau = 0.6$, $\tau = 0.8$ and $\tau = 1$

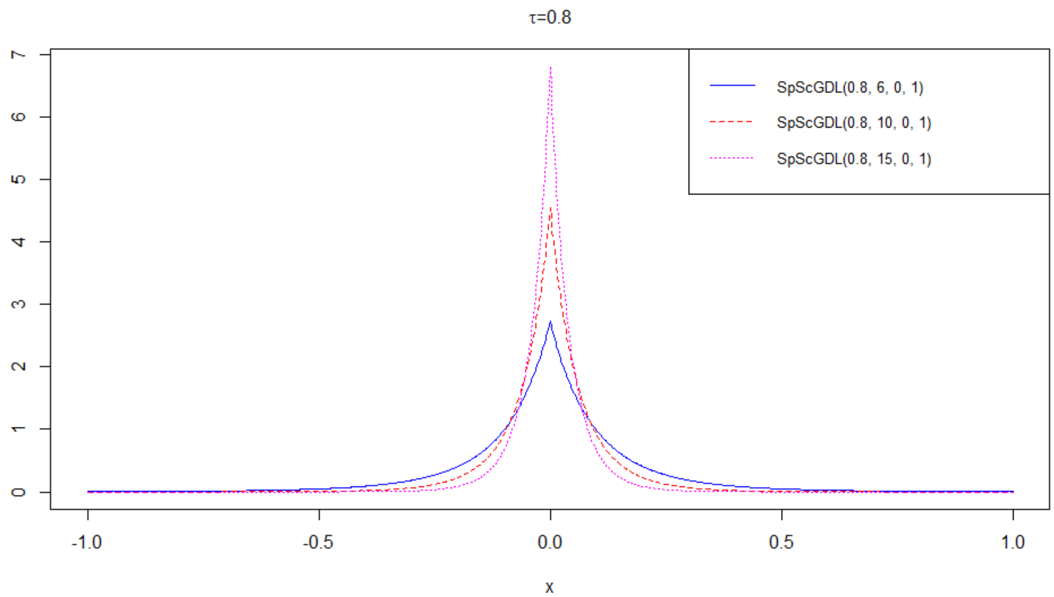

Figure 2. The densities of SpScGDL distributions with $v = 6$, $v = 10$ and $v = 15$

It is easy to see that the cdf of SpScGDL(τ, v, m, s) distribution is

$$F_{\mathrm{SpSc}}(x; \tau, v, m, s) = \begin{cases} (c_\tau \tau^2/2)\left(1 + \dfrac{m-x}{s\tau^2}\right)^{-v}, & \text{if } x < m; \\ 1 - \dfrac{c_\tau}{2\tau^2}\left(1 + \dfrac{\tau^2}{s}(x-m)\right)^{-v}, & \text{if } x \geq m, \end{cases}$$

where c_τ is defined in (1).

Properties of the spliced-scale generalized double Lomax distribution.

- **Unimodality.**

The spliced-scale generalized double Lomax distribution is unimodal and its mode equals m (the approach of Fernandez and Steel always yields a skewed unimodal distribution which mode coincides with the mode of the original symmetric distribution, see [7]).

- **Quantiles.**

It is easy to check that the following assertion holds.

Theorem 1. *The quantile functon* $Q_\tau(u)$ *of the* SpScGDL(τ, v, m, s) *distribution is given by*

$$Q_\tau(u) = \begin{cases} m - s\tau^2\left(\left(\dfrac{2u}{c_\tau \tau^2}\right)^{-1/v} - 1\right), & \textit{if } u \in \left(0; \dfrac{\tau^4}{1+\tau^4}\right]; \\ m + \dfrac{s}{\tau^2}\left(\left(\dfrac{2\tau^2(1-u)}{c_\tau}\right)^{-1/v} - 1\right), & \textit{if } u \in \left(\dfrac{\tau^4}{1+\tau^4}; 1\right). \end{cases}$$

- **Moments about the origin.**

Theorem 2. *The n-th moment about the origin of the* $\mathrm{SpScGDL}(\tau, v, m, s)$ *distribution equals*

$$\mu_n' = \frac{c_\tau v s^n}{2}\left((-1)^n \tau^{2n+2} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{a^k}{v-(n-k)} + \right. \\ \left. + \frac{1}{\tau^{2n+2}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{b^k}{v-(n-k)}\right), \tag{2}$$

where c_τ is defined by (1).

Proof. Indeed,

$$\mu_n' = I_1 + I_2,$$

where

$$I_1 = \int_{-\infty}^{m} x^n p_{\mathrm{SpSc}}(x) dx = c_\tau \frac{v}{2} (-1)^n s^n \tau^{2n+2} \int_{1}^{+\infty} \frac{(z-a)^n}{z^{v+1}} dz, \tag{3}$$

$a = 1 + \frac{m}{s\tau^2}$,

$$I_2 = \int_{m}^{\infty} x^n p_{\mathrm{SpSc}}(x) dx = c_\tau \frac{v}{2} s^n \frac{1}{\tau^{2n+2}} \int_{1}^{\infty} \frac{(z-b)^n}{z^{v+1}} dz, \tag{4}$$

$b = 1 - \frac{m\tau^2}{s}$.

Now (2) follows from (3) and (4). The theorem is proved.

- **Entropy.**

Theorem 3. *Let $\alpha > \frac{1}{v+1}$, $\alpha \neq 1$. The Renyi entropy H_α of* $\mathrm{SpScGDL}(\tau, v, m, s)$ *distribution equals*

$$H_\alpha = \ln \frac{2s}{c_\tau} + \frac{1}{1-\alpha} \ln \frac{v^\alpha}{\alpha(v+1)-1}.$$

Proof. We have:

$$H_\alpha = \frac{1}{1-\alpha} \ln \int_{\mathbb{R}} p_{\mathrm{SpSc}}^\alpha(x) dx.$$

Then

$$\int_{\mathbb{R}} p_{\mathrm{SpSc}}^\alpha(x) dx = \int_{-\infty}^{m} p_{\mathrm{SpSc}}^\alpha(x) dx + \int_{m}^{\infty} p_{\mathrm{SpSc}}^\alpha(x) dx,$$

$$\int_{-\infty}^{m} p_{\mathrm{SpSc}}^\alpha(x) dx = \left(c_\tau \frac{v}{2s}\right)^\alpha \cdot \frac{s\tau^2}{\alpha(v+1)-1},$$

$$\int_{m}^{\infty} p_{\mathrm{SpSc}}^\alpha(x) dx = \left(c_\tau \frac{v}{2s}\right)^\alpha \cdot \frac{s}{\tau^2(\alpha(v+1)-1)}.$$

Therefore

$$H_\alpha = \frac{1}{1-\alpha}\ln\left(\left(c_\tau\frac{v}{2s}\right)^\alpha \cdot \frac{s}{\alpha(v+1)-1}\cdot\left(\tau^2+\frac{1}{\tau^2}\right)\right) =$$
$$= \ln\frac{2s}{c_\tau} + \frac{1}{1-\alpha}\ln\frac{v^\alpha}{\alpha(v+1)-1}.$$

Theorem 4. *The Shannon entropy of the* SpScGDL(τ, v, m, s) *distribution equals*

$$H = \frac{v+1}{v} - \ln\frac{c_\tau v}{2s}. \tag{5}$$

Proof. We have

$$H = -\int\limits_{\mathbb{R}} p_{\text{SpSc}}(x)\ln p_{\text{SpSc}}(x)dx, \tag{6}$$
$$H = -(J_1+J_2),$$

where

$$J_1 = \frac{c_\tau v}{2s}\int\limits_{-\infty}^{m}\left(1+\frac{1}{\tau^2 s}(m-x)\right)^{-v-1}\ln\left(\frac{c_\tau v}{2s}\left(1+\frac{1}{\tau^2 s}(m-x)\right)^{-v-1}\right)dx, \tag{7}$$

$$J_2 = \frac{c_\tau v}{2s}\int\limits_{m}^{\infty}\left(1+\frac{\tau^2}{s}(x-m)\right)^{-v-1}\ln\left(\frac{c_\tau v}{2s}\left(1+\frac{\tau^2}{s}(x-m)\right)^{-v-1}\right)dx. \tag{8}$$

Equality (5) follows from (6), (7) and (8) after simple calculations. The theorem is proved.

Applications to real data.

The fit of the spliced-scale generalized double Lomax distribution was compared to fit of several other competing distributions using financial datasets. These datasets were the daily stock returns $\xi_k = \eta_{k+1} - \eta_k$ (where η_k is the stock price on day k) for the following stocks (see [8]; [9]):

- DHR, from January 19, 2017 to October 5, 2017;
- IFF, from June 20, 2003 to March 9, 2004.

The competing distributions were: the Johnson-S_U (JS$_U$) distribution, the sinh-arcsinh (SH-ASH) distribution (the "reparametrized" version, see [10], p. 768), the skew t (ST) distribution of Jones and Faddy (see [11]) and the normal inverse Gaussian (NIG) distribution.

The spliced-scale generalized double Lomax distribution was fitted using the maximum likelihood method. The numerical algorithm chosen for maximization of likelihood was the simulated annealing, the `R` programming language and the `R` package `optimization` were used.

The MLE estimates for the JS$_U$ distribution, the SH-ASH distribution and the ST distribution were obtained using `R` package `fitdistrplus`. The MLE estimates for the NIG distribution were procured by means of `R` package `GeneralizedHyperbolic`.

The AIC was taken as a goodness-of-fit statistic. The values of the AIC for the distributions fitted to the above-mentioned datasets are given in Table 1.

Table 1.

Values of AIC

DHR					
Distribution	SpScGDL	SH-ASH	JS_U	ST	NIG
AIC	347.463	349.884	349.312	349.614	349.929
IFF					
Distribution	SpScGDL	SH-ASH	JS_U	ST	NIG
AIC	165.796	168.055	166.123	167.124	167.022

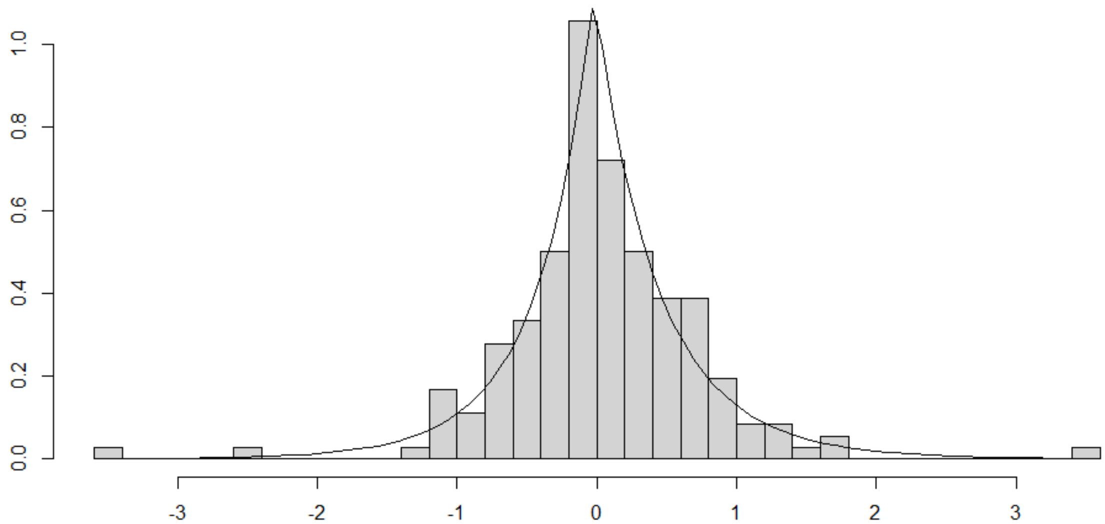

Figure 3. The histogram and the SpScGDL pdf for DHR dataset

The spliced-scale generalized double Lomax distribution corresponded to the lowest value of the AIC for both datasets.

Figure 3 shows the histogram and the fitted spliced-scale generalized double Lomax distribution pdf for DHR dataset.

3. Exponentiated double Lomax distribution..

The next skewed version of the generalized double Lomax distribution which will be considered is the exponentiated generalized double Lomax distribution. Skewing of a distribution family by creating exponentiated distributions is a well-known method. Namely, if F is a probability distribution with the cdf $F(x)$, then the corresponding exponentiated distribution F_γ is defined as the distribution with the cdf

$$F_\gamma(x) = (F(x))^\gamma,$$

where $\gamma \in (0; \infty)$.

General properties of exponentiated distributions and concrete examples of such distributions are given in detail in [6].

Definition 2. *The exponentiated generalized double Lomax distribution with parameters γ, v, m and s $(\gamma, v, s > 0)$ (or $\mathrm{EGDL}(\gamma, v, m, s)$ distribution) is defined as the distribution with the cdf*

$$F_\gamma(x) = (F(x))^\gamma,$$

where

$$F(x) = \begin{cases} \frac{1}{2}\left(1 + \dfrac{m-x}{s}\right)^{-v}, & \text{if } x < m; \\ 1 - \frac{1}{2}\left(1 + \dfrac{x-m}{s}\right)^{-v}, & \text{if } x \geq m, \end{cases} \tag{9}$$

is the cdf of the $\mathrm{GDL}(v, m, s)$ *distribution.*

Remark 2. *If* $\gamma = 1$ *then* $\mathrm{EGDL}(\gamma, v, m, s)$ *coincides with* $\mathrm{GDL}(v, m, s)$ *distribution.*

The pdf of $\mathrm{EGDL}(\gamma, v, m, s)$ distribution is

$$p_\gamma(x) = \gamma(F(x))^{\gamma-1}p(x), \tag{10}$$

where $p(x)$ is the pdf of the $\mathrm{GDL}(v, m, s)$ distribution. $p_\gamma(x)$ can also be represented as

$$p_\gamma(x) = \begin{cases} \gamma 2^{-\gamma}\frac{v}{s}\left(1 - \dfrac{x-m}{s}\right)^{-\gamma v - 1}, & \text{if } x < m; \\ \gamma \frac{v}{2s}\left(1 + \dfrac{x-m}{s}\right)^{-v-1}\left(1 - \frac{1}{2}\left(1 + \dfrac{x-m}{s}\right)^{-v}\right)^{\gamma-1}, & \text{if } x \geq m. \end{cases}$$

Figures 4 and 5 show several densities of exponentiated generalized double Lomax distributions.

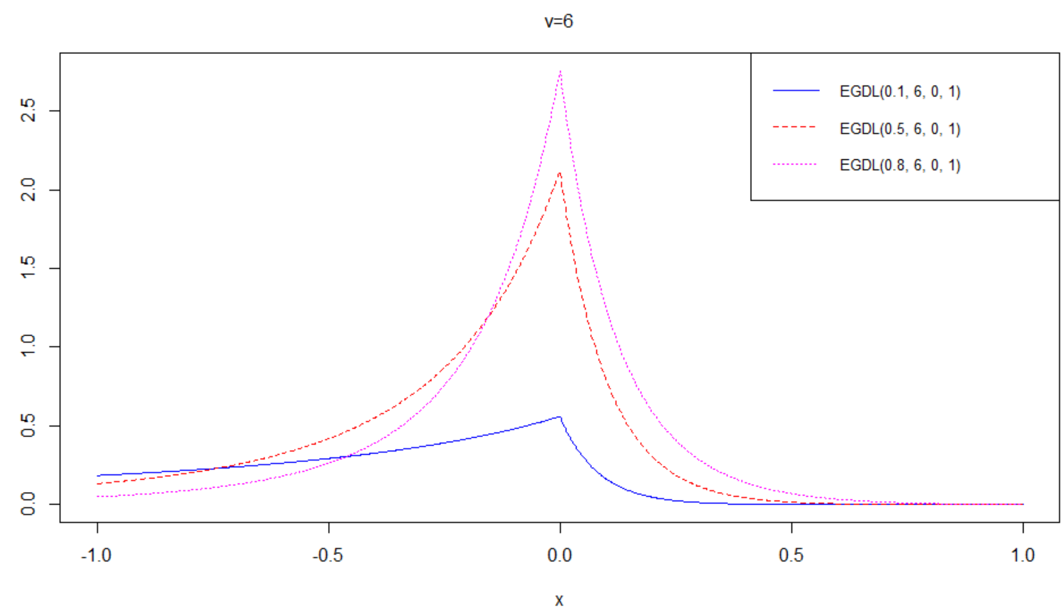

Figure 4. The densities of EGDL distributions with $\gamma = 0.1$, $\gamma = 0.5$ and $\gamma = 0.8$

Properties of the exponentiated generalized double Lomax distribution

- **Unimodality.**

Theorem 5. *The* $\mathrm{EGDL}(\gamma, v, m, s)$ *distribution is unimodal, its mode equals* m.

Proof. It is enough to prove that $p'_\gamma(x) > 0$ for $x < m$ and $p'_\gamma(x) < 0$ for $x > m$. We will establish (without loss of generality) this fact for the case $m = 0$.

Indeed, differentiating (10) we obtain

$$\begin{aligned} p'_\gamma(x) &= \gamma(\gamma-1)(F(x))^{\gamma-2}(p(x))^2 + \gamma(F(x))^{\gamma-1}p'(x) \\ &= \gamma(F(x))^{\gamma-2}\Big((\gamma-1)p^2(x) + F(x)p'(x)\Big), \end{aligned}$$

and therefore

$$\mathrm{sign}(p'_\gamma(x)) = \mathrm{sign}\Big((\gamma-1)p^2(x) + F(x)p'(x)\Big).$$

Let us consider the case $x < 0$. We have:

$$F(x)p'(x) = \frac{v(v+1)}{4s^2(1-x/s)^{2v+2}},$$

$$p^2(x) = \frac{v^2}{4s^2(1-x/s)^{2v+2}},$$

and $F(x)p'(x) > p^2(x)$ for $x < 0$. Therefore

$$\mathrm{sign}(p'_\gamma(x)) = \mathrm{sign}\Big((\gamma-1)p^2(x) + F(x)p'(x)\Big) > 0, \quad x < 0.$$

Inequality $p'_\gamma(x) < 0$ can be proved in a similar way for $x > 0$.

- **Quantiles.**

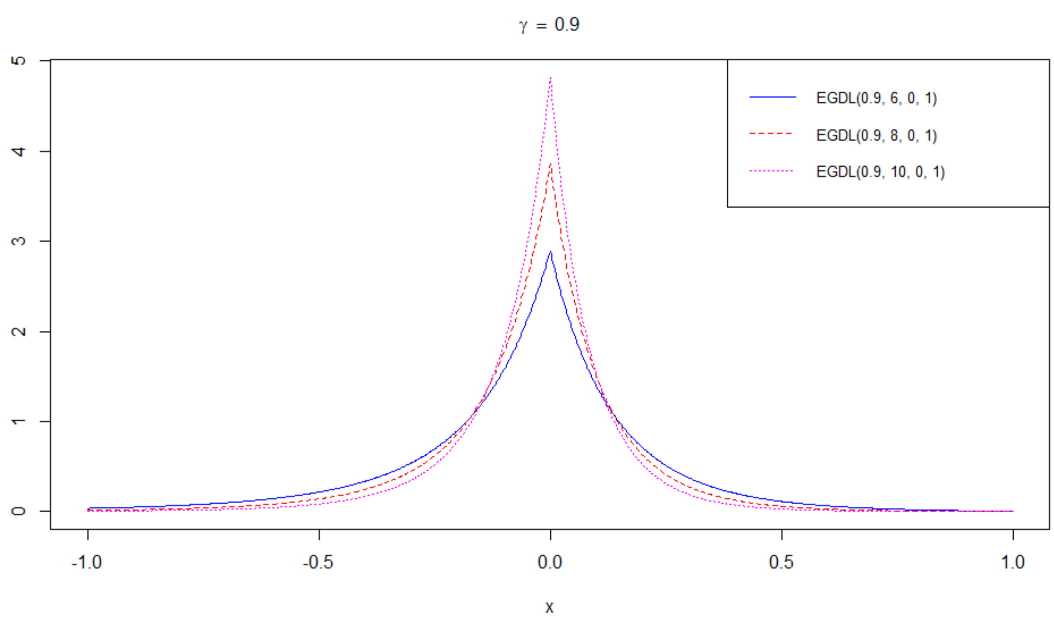

Figure 5. The densities of EGDL distributions with $v = 6$, $v = 8$ and $v = 10$

Theorem 6. *The quantile function* $Q_\gamma(u)$ *of the* EGDL(γ, v, m, s) *distribution is given by*

$$Q_\gamma(u) = \begin{cases} m + s\left(1 - 2^{-1/v} u^{-1/(\gamma v)}\right), & \text{if } u \in (0; 2^{-\gamma}]; \\ m + s\left(2^{-1/v}(1 - u^{1/\gamma})^{-1/v} - 1\right), & \text{if } u \in (2^{-\gamma}; 1). \end{cases} \tag{11}$$

Proof. (11) immediately follows from the equality

$$Q_\gamma\left((F(x))^\gamma\right) = x,$$

where $F(x)$ is the cdf of the GDL(v, m, s) distribution (see (9)).

Applications to real data

The fit of the exponentiated generalized double Lomax distribution was compared to fit of other competing distributions (which were already used for assessing the fit of spliced-scale generalized double Lomax distribution) using several stock datasets. The following daily stock returns were used (see [12]; [13]):

- BSET, from August 29, 2018 to March 22, 2019;
- REX, from May 6, 2004 to November 3, 2006.

The values of the AIC are given in Table 2.

Table 2.

Values of AIC

BSET					
Distribution	EGDL	SH-ASH	JS_U	ST	NIG
AIC	195.421	196.089	197.409	197.660	197.202
REX					
Distribution	EGDL	SH-ASH	JS_U	ST	NIG
AIC	193.487	198.660	203.294	206.033	201.564

Figure 6 shows the histogram and the fitted exponentiated generalized double Lomax distribution pdf for REX dataset.

The exponentiated generalized double Lomax distribution was fitted using the maximum likelihood method. Numerical maximization was accomplished using the simulated annealing method (`R` package `optimization`) for BSET dataset and the Hooke-Jeeves method (`R` package `dfoptim`) for REX dataset. Fitting of the competing distributions was done using the same `R` packages as for the SpScGDL distribution.

The exponentiated generalized double Lomax distribution corresponded to the lowest value of the AIC for both datasets.

4. Conclusions. Two new families of distributions were proposed: the spliced-scale generalized double Lomax distribution and the exponentiated generalized double Lomax distribution. These families can be successfully used for modeling heavy-tailed asymmetric data, e.g. stock returns.

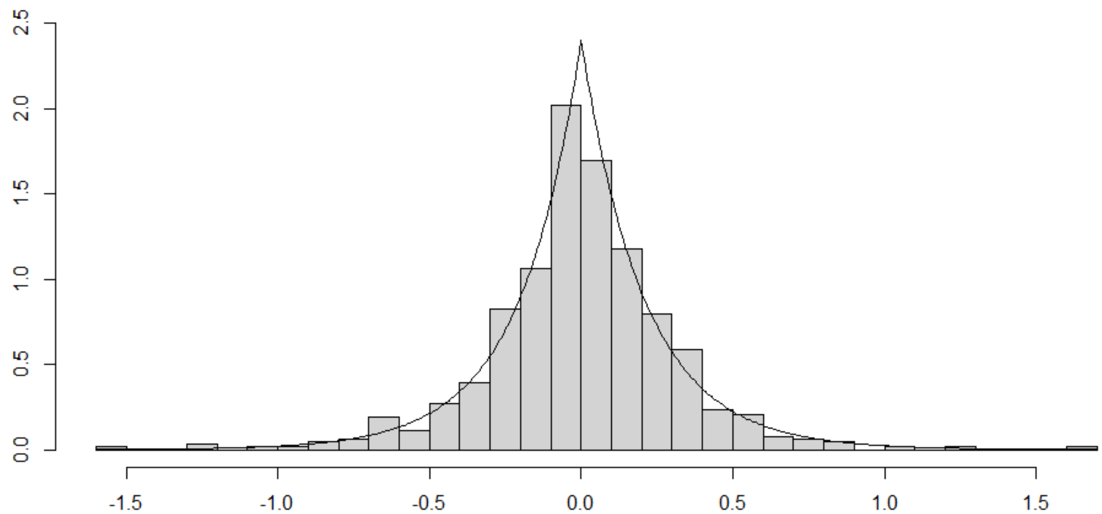

Figure 6. The histogram and the EGDL pdf for REX dataset

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Турчин Є. В. Асиметричні аналоги узагальненого подвійного розподілу Ломакса: властивості та застосування.

Розглянуто два аналоги так званого узагальненого подвійного розподілу Ломакса. Це узагальнений подвійний розподіл Ломакса із "кусково-сталим параметром масштабу" та "піднесений до ступеня" узагальнений подвійний розподіл Ломакса. Вивчені властивості цих розподілів. Придатність нових розподілів для реальних застосувань підтверджена їх підгонкою до наборів даних по приростам на ціни акцій.

Ключові слова: узагальнений подвійний розподіл Ломакса, метод максимальної правдоподібності, момент, ентропія, квантиль.

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