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THE DIFFRACTION OF ELASTIC WAVES BY SPHERICAL DEFECTS

Based on the method of discontinuous solutions [2–4] in the case of stationary elastic waves, a method is proposed for reducing a number of diffraction problems to a system of integro-differential equations. The defect can be either a spherical crack or a thin rigid spherical inclusion. Detailing of the method is considered for the second case.

Work goals. Generalization of the method of discontinuous solutions [2–4] to the case of spherical defects (cracks or thin rigid spherical inclusions). A method for constructing a discontinuous solution of the wave equation for a spherical coordinate system is proposed.

Keywords: wave equation, elasticity theory, defect, inclusion, crack, discontinuous solution, jump, spherical coordinates, stress, displacement.

1. Introduction. The study of the interaction of undeformed shells with the surrounding elastic medium is of practical importance, due to the need to increase the strength of ships from underwater and air explosions, improve the methods of underwater acoustics, and ensure the seismic resistance of hydraulic structures and their elements. Thus, the development of mathematical methods for solving problems on the interaction of non-stationary (stationary) waves with various objects, including shell type, is relevant.

Among the analytical methods, the following can be distinguished: the method of integral equations (the potential method), the method of separation of variables and its various modifications (the Fourier method and its generalizations in vector and scalar forms, as well as reduction to infinite systems of algebraic equations), the method of the theory of functions of a complex variable. These methods have proven themselves well in relation to canonical domains (the equations of their boundary surfaces are reduced to standard canonical forms). The following authors were closely involved in this topic: Guz' A. N., Nemish Yu. N., Kubenko Yu. N., Podstrigach Ya. S., Grilitsky D. V., Poddubnyak A. P. and etc.

At present, various numerical methods of finite differences, finite elements, etc. are widely used to solve spatial problems. The proposed work is devoted to solving a spatial problem of elasticity theory for a spherical segment by the method of discontinuous solutions [2–4].

2. Main results.

Part 1. Construction of a discontinuous solution of the wave equation for a spherical defect

Under the defect (from the point of view of mechanics) we mean [4] a part of the surface, at the intersection of which the stresses and displacements of the first kind suffer discontinuities. As a classical defect, we can consider some mathematical cut along the specified part of the surface (crack). A certain rigid inclusion in the form of a shell (cavity), the middle surface of which coincides with the same part of the surface, can also be attributed to such defects. Consider, as one of the special cases, when a part of a spherical surface serves as a defect.

Let's set its geometric parameters in the form: $r = R$, $0 \le \theta \le \omega$, $-\pi \le \varphi \le \pi$, where r, θ, φ are the parameters of the spherical coordinate system. It is widely known that the solution of the equations of motion of an elastic isotropic medium can be expressed in terms of wave functions [1]. Therefore, before proceeding with the construction of a discontinuous solution for the equations of motion, one should construct a solution for the wave equation

$$\triangle\, \psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = 0, \quad 0 < r < \infty,\ 0 < \theta < \pi,\ |\varphi| < \pi,\ t \ge 0, \tag{1}$$

where $\triangle$ is the Laplace operator expressed in spherical coordinates.

Under the discontinuous solution of equation (1), which is given in the entire space for a spherical defect

$$r = R, \quad 0 \le \theta \le \omega, \quad -\pi \le \varphi \le \pi \tag{2}$$

one should understand such a solution to equation (1), which must satisfy it everywhere, excluding only the points of the defect itself (2) (R is the radius of a spherical defect). At these points, the function and its normal (to the surface of the considered defect) derivative suffer discontinuities of the first kind and their jumps are given, for which we introduce special notation

$$\psi(R-0,\theta,\varphi,t) - \psi(R+0,\theta,\varphi,t) = \langle\psi\rangle,$$

$$\psi'(R-0,\theta,\varphi,t) - \psi'(R+0,\theta,\varphi,t) = \langle\psi'\rangle.$$

In addition, here and everywhere below in the text we will denote the derivative with respect to the variable r by a prime, with respect to θ by a dot, and with respect to the variable φ by a comma. To construct such a solution, we use the same scheme as in the materials [4].

By successively applying to equation (1) the integral transformations of Laplace (with respect to the variable t), Fourier (with respect to the variable φ)

$$\psi_p = \int_0^\infty \frac{\psi(r,\theta,\varphi,t)}{e^{pt}}dt, \qquad \psi_{pn} = \int_{-\pi}^{\pi} \frac{\psi_p(r,\theta,\varphi)}{e^{in\varphi}}d\varphi, \tag{3}$$

and Legendre (with respect to the variable θ),

$$\psi_{pnk}(r) = \int_0^\pi \sin\theta P_k^{|n|}(\cos\theta)d\theta, \tag{4}$$

($P_k^n(\cos\theta)$ is the adjointed Legendre polynomial), we reduce equation (1) to the following one-dimensional form

$$\frac{1}{r^2}\left[\left(r^2\psi'_{pnk}(r)\right)' - k(k+1)\psi_{pnk}(r)\right] - \frac{p^2}{c^2}\psi_{pnk}(r) = 0, \tag{5}$$

where $0 < r < \infty$.

At this stage, it is necessary to construct a discontinuous solution of this equation with predetermined jumps

$$\begin{aligned} \langle\psi_{pnk}\rangle &= \psi_{pnk}(R-0) - \psi_{pnk}(R+0), \\ \langle\psi'_{pnk}\rangle &= \psi'_{pnk}(R-0) - \psi'_{pnk}(R+0). \end{aligned} \tag{6}$$

The values of these jumps will be determined based on the boundary conditions of the problem.

If in (5) we make a change of variables of the form $\chi_{pnk}(r) = \sqrt{r}\ \psi_{pnk}(r)$, then this equation is transformed into the Bessel equation. Let us apply the Hankel transformation to the resulting equation

$$\chi_{pnk}(r) = \int_0^\infty r J_{k+\frac{1}{2}}(\alpha r)\chi_{pnk}(r)dr,$$

to get rid of the variable r according to the generalized scheme [2,3] (in this formula, $J_{k+\frac{1}{2}}(\alpha r)$ is the cylindrical Bessel function).

Using the obtained results, we find the dimensionless Hankel transform from the equation (5), expressing them in terms of jumps (6). Further, applying to this expression the inversion formula for the Hankel transform, we find the necessary discontinuous solution of equation (5) with jumps (6)

$$\begin{gathered} \psi_{pnk}(r) = R^2\left[\langle\psi'_{pnk}\rangle D_{k,p}(r,R) - \langle\psi_{pnk}\rangle\frac{\partial}{\partial R}D_{k,p}(r,R)\right], \\ D_{k,p}(r,R) = \frac{1}{\sqrt{rR}}\begin{cases} I_\nu\left(\frac{Rp}{c}\right)K_\nu\left(\frac{rp}{c}\right), & r > R, \\ I_\nu\left(\frac{rp}{c}\right)K_\nu\left(\frac{Rp}{c}\right), & r < R,\ \nu = k+\frac{1}{2},\ k = 0,1,2,\ldots \end{cases} \end{gathered} \tag{7}$$

($I_\nu(z)$, $K_\nu(z)$ are respectively modified Bessel and Macdonald functions). Further, to obtain a discontinuous solution of the original wave equation, one should use the inversion formulas for the Legendre transforms [2, 3]

$$\psi_{pn}(r,\theta) = \sum_{k=|n|}^\infty \psi_{pnk}(r)\sigma_{kn}P_k^{|n|}(\cos\theta), \quad \sigma_{kn} = \frac{(k-|n|)!(2k+1)}{2(k+|n|)!}, \tag{8}$$

as well as for the Fourier and Laplace transforms.

Thus, applying transformation (8) to formula (7), we obtain the following equation

$$\psi_{pn}(r,\theta) = R^2\left[\int_0^{\omega} T_{n,p}(\theta,\tau)\sin\tau d\tau - \int_0^{\omega}\widetilde{T}_{n,p}(\theta,\tau)\sin\tau d\tau\right],$$
$$T_{n,p}(\theta,\tau) = \langle\psi'_{pn}\rangle M_{n,p}(\theta,\tau,r,R), \qquad \widetilde{T}_{n,p}(\theta,\tau) = \langle\psi_{pn}\rangle\frac{\partial}{\partial R}M_{n,p}(\theta,\tau,r,R), \tag{9}$$
$$M_{n,p}(\theta,\tau,r,R) = \sigma_{kn}P_k^{|n|}(\cos\theta)P_k^{|n|}(\cos\tau)D_{k,p}(r,R).$$

In the event that a steady process of medium oscillations is considered (occurring according to a harmonic law), then the potential from the wave equation (1) can be written in the following form

$$\psi\left(r,\theta,\varphi,t\right) = e^{-i\omega_0 t}\widetilde{\psi}\left(r,\theta,\varphi\right). \tag{10}$$

This makes it possible to exclude the use of the direct and inverse Laplace transforms with respect to the variable t, which greatly simplifies the calculations. Then, if in equation (5), instead of the parameter p, we substitute the value $p = -i\omega_0$, then we obtain a new equation, which is the solution for the function $\widetilde{\psi}\left(r,\theta,\varphi\right)$.

In contrast to equation (7), the discontinuous solution in this case will take a slightly different form

$$\widetilde{\psi}_{pnk}(r) = R^2\left[\langle\psi'_{pnk}\rangle D_{k,\mu}(r,R) - \langle\psi_{pnk}\rangle\frac{\partial}{\partial R}D_{k,\mu}(r,R)\right], \tag{11}$$

$$D_{k,\mu}(r,R) = \frac{\pi i}{2\sqrt{rR}}\begin{cases} J_\nu(R\mu)H_\nu^{(1)}(r\mu), \ r>R, \ \mu=\frac{\omega_0}{c}, \\ J_\nu(r\mu)H_\nu^{(1)}(R\mu), \ r<R, \ \nu=k+\frac{1}{2}, \ k=0,1,2,\ldots \end{cases}$$

If in (11) we invert the Legendre transforms, then we obtain an equation of the following form

$$\widetilde{\psi}_n(r,\theta) = R^2\left[\int_0^{\omega} P_{n,\mu}(\theta,\tau)\sin\tau d\tau - \int_0^{\omega}\widetilde{P}_{n,\mu}(\theta,\tau)\sin\tau d\tau\right],$$
$$P_{n,\mu}(\theta,\tau) = \langle\widetilde{\psi}'_{pn}\rangle M_{n,\mu}\left(\theta,\tau;r,R\right), \quad \widetilde{P}_{n,\mu}(\theta,\tau) = \langle\widetilde{\psi}_{pn}\rangle\frac{\partial}{\partial R}M_{n,\mu}\left(\theta,\tau;r,R\right), \tag{12}$$
$$M_{n,\mu}\left(\theta,\tau;r,R\right) = \sigma_{kn}P_k^{|n|}(\cos\theta)P_k^{|n|}(\cos\tau)D_{k,\mu}(r,R).$$

When substituting the value $p = -i\omega_0$ in (7), it is necessary to choose the first Hankel function $H_\nu^{(1)}\left(z\right)$ in the kernel $D_{k,\mu}\left(r,R\right)$. It is she who provides the condition of radiation at infinity. The second function $J_\nu\left(z\right)$ in this kernel is the cylindrical Bessel function. When using discontinuous solutions of the form (9) and (12) in specific problems of the theory of elasticity, it is necessary to use the integral representation for the following function

$$W_k(z)|_{z=-i\xi} = I_\nu(z)K_\nu(z)|_{z=-i\xi} = \frac{\pi i}{2}H_\nu^{(1)}(\xi)J_\nu(\xi) = A_k(\xi), \quad \nu = k+\frac{1}{2}. \tag{13}$$

To obtain relation (13), it suffices to use formula, which allows us to expand the functions $\Omega_0(\theta) = I_0(\theta) - L_0(\theta)$ ($L_0(\theta)$ — the second Struve function [2]) into a series in the orthogonal system of functions $\cos\left[\left(k+\frac{1}{2}\right)\theta\right]$ and therefore

$$W_k(z) = \frac{(-1)^k}{2}\int_0^\pi \Omega_0\left(2z\cos\frac{\theta}{2}\right)\cos\left[\left(k+\frac{1}{2}\right)\theta\right]d\theta.$$

Integrating by parts based on (13), we establish an important relationship:

$$A_k(\xi) = \frac{1-\Delta_k(\xi)}{2k+1}, \quad \Delta_k(\xi) = \int_0^\pi \frac{\sin[(k+\frac{1}{2})\tau]}{(-1)^k}\frac{\partial}{\partial\tau}\Upsilon_0\left(2\xi\cos\frac{\tau}{2}\right)d\tau, \tag{14}$$

where $\Upsilon_0(z) = J_\nu(z) - iH_0(z)$ is the first Struve function.

Part 2. Construction of a discontinuous solution of the equations of motion of an elastic medium for a spherical defects

In order to construct a discontinuous solution to the equations of motion of an elastic medium, we use the well-known solution to the equations of motion of an isotropic medium, which, following the notation of the authors, is expressed in terms of three wave functions $\Phi(r,\theta,\varphi,t)$, $\Psi_j(r,\theta,\varphi,t)$ $(j=1,2)$. It should be noted that the function $\Phi(r,\theta,\varphi,t)$ determines the expansion wave and must satisfy formula (1), in which one should put $c=c_1$, where c_1 is the speed of the expansion wave [1]. In turn, the functions $\Psi_j(r,\theta,\varphi,t)$ $(j=1,2)$ describe shear waves and in the same equation one should put $c=c_2$, where c_2 is the speed of shear wave propagation. If we omit the time parameter t, that is, we restrict ourselves to a simpler case when a steady process of oscillations according to a harmonic law with a certain natural oscillation frequency ω_0 is considered, then, according to [1], the wave potentials can be represented as $\{\Phi,\Psi_j\} = e^{-i\omega_0 t}\left\{\widetilde{\Phi},\ \widetilde{\Psi}_j\right\}$, respectively.

Using the materials from [2–4], we pass everywhere to the Fourier transforms

$$\{\Phi_n, \Psi_{j,n}, u_{r,n}, u_{\theta,n}, u_{\varphi,n}\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\omega_0 t}\{\Phi,\Psi,u_r,u_\theta,u_\varphi\}\,d\varphi, \tag{15}$$

$(j=1,2,\ n=0,\pm1,\pm2,\ldots)$.

If in formula (15) we formally omit the factor depending on the variable t $(e^{-i\omega_0 t})$, as well as the designation of the wave above the symbols, the solutions of the above equations can be represented in the following form

$$\begin{aligned} u_n \equiv u_{r,n} &= \Phi_n' - \frac{[\sin\theta\Psi_{2,n}]^\bullet}{r\sin\theta} - \frac{n^2\Psi_{2,n}}{r\sin^2\theta}, \\ v_n \equiv u_{\theta,n} &= \frac{\Psi_n^\bullet}{r} + \frac{(r\Psi_{2,n})'}{r} + \frac{in\Psi_{1,n}}{\sin\theta}, \\ w_n \equiv u_{\varphi,n} &= \frac{in\Phi_n}{r\sin\theta} + \frac{in\,(r\Psi_{2,n})'}{r\sin\theta} - \Psi_{1,n}^\bullet, \end{aligned} \tag{16}$$

here and everywhere below, as in [2], we denote the derivative with respect to the variable r by a prime, with respect to θ by a dot, and with respect to the variable

φ by a comma. It should be noted that the functions Φ_n and $\Psi_{j,n}$ $(j=1,2)$, must satisfy the Helmholtz equation. For clarity, combining them into a single equation, we write

$$\frac{1}{r^2}\left\{\left(r^2\left[\Phi_n', \Psi_{j,n}'\right]\right)' - \nabla_n\left[\Phi_n, \Psi_{j,n}\right]\right\} + \left[a^2\Phi_n,\ b^2\Psi_{j,n}\right] = 0, \quad (j=1,2), \tag{17}$$

$$\nabla_n f(r,\theta) \equiv \frac{n^2 f(r,\theta)}{\sin^2\theta} - \frac{\left[\sin\theta f^{\bullet}(r,\theta)\right]^{\bullet}}{\sin\theta}, \tag{18}$$

where $a = \frac{\omega_0}{c_1},\ b = \frac{\omega_0}{c_2}$.

Applying Hooke's law and the Cauchy relations to the displacement transformants (16), we find the stress transformants necessary for further calculations

$$\frac{\sigma_{r,n}}{2\mu} = \Phi_n'' - \lambda\frac{a^2\Phi_n}{2\mu} + b^2\left(\Psi_{2,n} + r\Psi_{2,n}'\right) + 3\Psi_{2,n}'' + r\Psi_{2,n}''',$$

$$\frac{\tau_{\theta,n}}{2\mu} = \frac{\Phi_n'}{r} - \frac{\Phi_n^{\bullet}}{r^2} - \frac{inr}{2\sin\theta}\left(\frac{\Psi_{1,n}}{r}\right)' + \Psi_{2,n}'' + \frac{\Psi_{2,n}'}{r} - \frac{\Psi_{2,n}^{\bullet}}{r^2} + \frac{b^2\Psi_{2,n}^{\bullet}}{2}, \tag{19}$$

$$-\frac{\tau_{\varphi,n}}{2\mu} = \frac{in\left(\Phi_n' - \frac{\Phi_n}{r}\right)}{r\sin\theta} + \frac{r}{2}\left(\frac{\Psi_{1,n}^{\bullet}}{r}\right)' + \frac{in\left(\Psi_{2,n}'' + \frac{\Psi_{2,n}'}{r} - \left(\frac{1}{r^2} - \frac{b^2}{2}\right)\Psi_{2,n}\right)}{\sin\theta},$$

where $\mu,\ \lambda$ are Lame parameters, $\tau_{\theta,n},\ \tau_{\varphi,n}$ are Fourier transforms for tangential stresses $\tau_{r,\theta}$ and $\tau_{r,\varphi}$ respectively.

A discontinuous solution of the equations of motion of an elastic medium for spherical defects (formula (2)) should be understood as such a solution of the above equations, which must satisfy them everywhere, except for the points of the defect. At these points, all components of the displacement and stress field suffer discontinuities of the first kind with given jumps. The values of these jumps are determined from the boundary conditions. Let us introduce the following notation for these jumps in terms of the Fourier transforms

$$\langle u_n\rangle = u_{rn}(R-0,\ \theta) - u_{rn}(R+0,\ \theta),\ \langle v_n\rangle,\ \langle w_n\rangle, \langle\sigma_{r,n}\rangle,\ \langle\tau_{\theta,n}\rangle, \langle\tau_{\varphi,n}\rangle, \tag{20}$$

where R is the radius of the spherical defect.

Further construction of a discontinuous solution will continue according to the scheme of works [4]. At the first stage of calculations, instead of the values $v_n,\ w_n$, $\tau_{\theta,n},\ \tau_{\varphi,n}$, for convenience, we should introduce their combinations

$$\begin{aligned} \sin\theta\zeta_n(r,\theta) &= \left[\sin\theta v_n(r,\theta)\right]^{\bullet} - inw_n(r,\theta), \\ \sin\theta\zeta_n(r,\theta) &= \left[\sin\theta w_n(r,\theta)\right]^{\bullet} + inv_n(r,\theta), \\ \sin\theta\vartheta_n(r,\theta) &= \left[\sin\theta\tau_{\theta,n}(r,\theta)\right]^{\bullet} + in\tau_{\varphi,n}(r,\theta), \\ sin\theta\varrho_n(r,\theta) &= \left[\sin\theta\tau_{\varphi,n}(r,\theta)\right]^{\bullet} + in\tau_{\theta,n}(r,\theta). \end{aligned} \tag{21}$$

Based on the notation introduced in (21), from relations (16), taking into account (18), we obtain fairly compact expressions for the following elements

$$\begin{aligned} u_n(r,\theta) &= \Phi_n'(r,\theta) + \tfrac{\nabla_n\Psi_{2,n}\ (r,\theta)}{r}, \\ \xi_n(r,\theta) &= \nabla_n\Psi_{1,n}\ (r,\theta), \\ r\zeta_n(r,\theta) &= -\nabla_n\left[\Phi_n(r,\theta) + \left(r\Psi_{2,n}\ (r,\theta)\right)'\right]. \end{aligned} \tag{22}$$

At this stage, according to [4], it is necessary to express all unknown jumps of the functions $\langle\Phi_n\rangle$, $\langle\Phi'_n\rangle$, $\langle\Psi_{j,n}\rangle$, $\langle\Psi'_{j,n}\rangle$ $(j=1,2)$ through the given (based on the boundary conditions of the problem posed) (6) or jumps of the introduced combinations of functions (21). Passing in formulas (16) to jumps and carrying out the necessary transformations in order to obtain jumps of functions (21), we obtain the following, very simple expressions for them

$$\begin{gathered}\langle u_n\rangle = \langle\Phi'_n\rangle + \tfrac{\nabla_n\langle\Psi_{2,n}\rangle}{R}, \quad \langle\xi_n\rangle = \nabla_n\langle\Psi_{1,n}\rangle, \\ R\langle\zeta_n\rangle = -\nabla_n\left[\langle\Phi_n\rangle + \langle\Psi_{2,n}\rangle + R\langle\Psi'_{2,n}\rangle\right].\end{gathered} \tag{23}$$

In order to obtain the same ratios for stresses, it is necessary in formulas (5) to exclude the terms that contain derivatives with respect to the variable r above the first order. To do this, use the group of equations (3), which will allow us to write the following relations

$$\Phi''_n = \frac{\nabla_n\Phi_n}{r^2} - \frac{2\Phi'_n}{r} - a^2\Phi_n,$$

$$\Psi''_{j,n} = \frac{\nabla_n\Psi_{j,n}}{r^2} - \frac{2\Psi'_{j,n}}{r} - b^2\Psi_{j,n} \quad (j=1,2).$$

Eliminating the indicated derivatives from (19) with the help of these formulas and passing in them to stress jumps (6), including jumps ϑ_n and ϱ_n, we obtain the following relations

$$\begin{gathered}\tfrac{\langle\sigma_{r,n}\rangle}{2\mu}R^2 = \nabla_n\left[\langle\Phi_n\rangle - \langle\Psi_{2,n}\rangle\right] - \tfrac{(bR)^2}{2}\langle\Phi_n\rangle - 2R\langle\Phi'_n\rangle + 2R\nabla_n\langle\Psi'_{2,n}\rangle, \\ \tfrac{R\langle\varrho_n\rangle}{\mu} = \nabla_n\left[R\langle\Psi'_{1,n}\rangle - \langle\Psi_{1,n}\rangle\right], \\ \tfrac{2R^2\langle\vartheta_n\rangle}{\mu} = -\nabla_n\left\{R\langle\Phi'_n\rangle - \langle\Phi_n\rangle + \nabla_n\langle\Psi_{2,n}\rangle - \langle\Psi_{2,n}\rangle\left[\tfrac{b^2}{2}+1\right] - R\langle\Psi'_{2,n}\rangle\right\}.\end{gathered} \tag{24}$$

At this stage, it is necessary to express all unknown jumps of wave functions and their normal derivatives through jumps of displacements and stresses, which can be determined from the boundary conditions and, in fact, are known. To do this, apply to all formulas (23) and (24) the Legendre integral transformation, according to formula (4). After that, as a result of fairly obvious transformations, we obtain the following expressions

$$\begin{gathered}k(k+1)\langle\Psi_{1,n,k}\rangle = \langle\xi_{n,k}\rangle, \\ k(k+1)\langle\Psi'_{1,n,k}\rangle = \tfrac{\langle\xi_{n,k}\rangle}{R} + \tfrac{\langle\varrho_{n,k}\rangle}{\mu}, \\ k\,(k+1)\,b^2R\langle\Psi_{2,n,k}\rangle = \tfrac{R\langle\vartheta_{n,k}\rangle}{\mu} + 2k\,(k+1)\,\langle u_{n,k}\rangle + 2\langle\zeta_{n,k}\rangle, \\ -Rb^2\langle\Phi_{n,k}\rangle = 4\langle u_{n,k}\rangle + 2\langle\zeta_{n,k}\rangle + \tfrac{R\langle\sigma_{r,n,k}\rangle}{\mu}, \\ (Rb)^2\,\langle\Phi'_{n,k}\rangle = \left[(Rb)^2 - 2k(k+1)\right]\langle u_{n,k}\rangle - \tfrac{R\langle\vartheta_{n,k}\rangle}{\mu} - 2\langle\zeta_{n,k}\rangle, \\ k\,(k+1)\,(Rb)^2\,\langle\Psi'_{2,n,k}\rangle = 2\langle u_{n,k}\rangle k(k+1) - \tfrac{R\langle\vartheta_{n,k}\rangle}{\mu} + \\ +\langle\zeta_{n,k}\rangle\left[2k\,(k+1) - 2 - (Rb)^2\right] + \tfrac{k(k+1)R\langle\sigma_{r,n,k}\rangle}{\mu}.\end{gathered} \tag{25}$$

According to formula (11), the Fourier-Legendre transforms of the wave functions will be expressed by the formulas

$$\begin{gathered}\Phi_{n,k}(r) = R^2\left[\langle\Phi'_{n,k}\rangle D_{k,\mu}(r,R) - \langle\Phi_{n,k}\rangle \tfrac{\partial}{\partial R} D_{k,\mu}(r,R)\right], \\ \Psi_{j,n,k}(r) = R^2\left[\langle\Psi'_{j,n,k}\rangle D_{k,\mu}(r,R) - \langle\Psi_{j,n,k}\rangle \tfrac{\partial}{\partial R} D_{k,\mu}(r,R)\right] \ (j=1,2).\end{gathered} \tag{26}$$

Substituting the jump values (11) into these formulas and then inverting the Legendre transformation, according to formula (8), we obtain the functions Φ_n and $\Psi_{j,n}$ $(j=1,2)$. Further, using the groups of formulas (16), (19) and the obtained wave potentials, we construct a discontinuous solution of the equations of motion for a spherical defect (2). Having a discontinuous solution, it is not difficult to reduce the problem of diffraction by such a defect to one-dimensional integral or integro-differential equations.

3. Conclusions. In the proposed work, a discontinuous solution of the wave equation is constructed in a spherical coordinate system. Based on the same method, a discontinuous solution of the equations of motion of an elastic medium for a spherical defect is constructed.

At the next stage, the problem of diffraction of an elastic torsion wave by a thin spherical inclusion should be reduced to a system of integro-differential equations.

Develop and prove the validity of using an approximate method for solving the corresponding integro-differential equations in the class of functions with non-integrable singularities.

Numerically implement the method, build graphs of the dependence of the reactive torque (the inclusion is fixedly fixed) on the frequency of oscillations and the dimensions of the inclusion. Also, build graphs for the amplitude of oscillations of the inclusion when it is mobile (not fixed).

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Назаренко О. А., Стехун А. О., Яровий А. Т. Дифракція пружних хвиль на сферичних дефектах.

На основі методу розривних рішень [2–4] у разі стаціонарних пружних хвиль запропоновано метод зведення ряду задач дифракції до системи інтегро-диференціальних рівнянь. Дефектом може бути як сферична тріщина, або тонке жорстке сферичне включення.

Деталізація методу розглядається для другого випадку. Узагальнення методу розривних розв'язків [2–4] на випадок сферичних дефектів (тріщин або тонких жорстких сферичних включень). Запропоновано метод побудови розривного розв'язку хвильового рівняння для сферичної системи координат.

Ключові слова: хвильове рівняння, теорія пружності, дефект, включення, тріщина, розривний розв'язок, стрибок, сферичні координати, напруження, переміщення.

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