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ON THE AUSLANDER ALGEBRA OVER A FIELD OF CHARACTERISTIC TWO OF THE COMMUTATIVE NONCYCLIC SEMIGROUP OF THIRD ORDER WITHOUT UNIT AND ZERO ELEMENTS

The classification of the semigroups of third order (in terms of Cayley tables, up to isomorphism and antiisomorphism) was first received by T. Tamura in 1953, and later, but with the help of a computer program, by G. E. Forsyth (in 1955). The minimal systems of generators and the corresponding defining relations for all such semigroups were constructed in the papers of V. M. Bondarenko and Y. V. Zatsikha. They also described representation type of third-order semigroups over an arbitrary field and in the case of semigroups of finite representation type, the canonical forms of matrix representations were indicated.

In a number of previous papers, the authors studied matrix Auslander algebras for third-order semigroups. This paper continues such research.

Keywords: semigroup, antiisomorphism, minimal system of generators, defining relations, matrix representation, representation type, canonical form, Auslander algebra.

1. Introduction. Minimal systems of generators and corresponding defining relations for all third-order semigroups are described in [1]. If we consider only commutative semigroups, and also those that are not neither cyclic nor cyclic with an attached unit or zero element, then there exist, up to isomorphism, only four semigroups (in square brackets are indicated all elements, in angular a minimal system of generators, and then the defining relations):

(a) $(0, b, c) = \langle b, c\rangle$: $b^2 = 0$, $c^2 = 0$, $bc = cb = 0$;

(b) $(0, b, c) = \langle b, c\rangle$: $b^2 = b$, $c^2 = c$, $bc = cb = 0$;

(c) $(0, b, c) = \langle b, c\rangle$: $b^2 = 0$, $c^2 = c$, $bc = cb = 0$;

(d) $(c^2, b, c) = \langle b, c\rangle$: $b^3 = b^2$, $c^3 = c$
(is a consequence of the remaining relations),
$b^2 = c^2$, $bc = cb = c$.

Note that trivial relations for unit and zero generating 0 and e (if any) are not written out.

All these semigroups are tame; moreover, except for the semigroup (a), are of finite representation type [1], that is, have a finite number of equivalence classes of indecomposable representations.

In the case of finite representation types, one of the forms of studying the category of representations is the description of the Auslander algebra as the algebra of endomorphisms of the direct sum of representatives of all equivalence classes of indecomposable representations. In the simple case of (b), the Auslander algebra was considered as an example in the paper [2], and in the case of (c) it was studied in [3]. In [4], we study the Auslander algebra of the semigroup (d) over a field of characteristic not equal to 2. The remaining case (when the characteristic is equal to 2) is considered in this paper.

2. Formulation of the main result. Let S be a semigroup and $T = \{T(x) \mid x \in S\}$ be its matrix representation over a field K. An *endomorphism of the representation* T is a matrix X such that $T(x)X = XT(x)$ for any $x \in S$. It is clear that when a system of generating elements of S is fixed, then it is sufficient to consider the indicated equality only for such elements. All matrices with this property form an algebra called the *algebra of endomorphisms of* T. In the case when the semigroup S has a finite representation type, the algebra of endomorphisms of the direct sum of all, with accuracy up to equivalence, indecomposable representations (i.e., one representative from each equivalence class) is called the *Auslander algebra of the semigroup* S *over the field* K and is denoted by $Aus_K(S)$. Since we are considering matrix representations, it is natural to call the *Auslander matrix algebra.* Note that it does not depend on the choice of representatives in the equivalence classes in the sense that all algebras obtained in this way are isomorphic; moreover, they are conjugated as subalgebras of the complete matrix algebra of the corresponding size.

We now formulate the main result. The semigroup of the form (d) is denoted by S_d.

Theorem 1. *The Auslander matrix algebra $Aus_K(S_d)$ of the semigroup S_d over a field K of characteristic* 2 *consists of all matrices of the form*

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & 0 & 0 & 0 \\ 0 & x_{11} & 0 & 0 & 0 & 0 \\ 0 & x_{32} & x_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{44} & x_{45} & x_{46} \\ 0 & 0 & 0 & 0 & x_{44} & 0 \\ 0 & 0 & 0 & 0 & x_{65} & x_{66} \end{pmatrix},$$

where x_{ij} are elements of K.

3. Proof of Theorem 1. Canonical forms of the matrix representations of the semigroups of third order over a field K were obtained in [1], using methods of the Kyiv school on the theory of matrix problems and representations (see, e.g., [5]–[16]). In particular, for the semigroup S_d in the case of $\operatorname{char} K = 2$, the canonical form looks like this:

$$b \to B = \begin{pmatrix} E & 0 & 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad c \to C = \begin{pmatrix} E & E & 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (*)$$

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where E are some unit cells (the size of which we do not fix). Since any representation is equivalent to an representation of the form $(*)$ (at the same time, some unit cells may be empty), then this property holds for the direct sum Σ of all representatives of equivalence classes of indecomposable representations. So it is possible to calculate the Auslander algebra assuming that the representation Σ has the form $(*)$. And it is clear that at the same time (if we reason purely formally) all unit cells have dimension 1 or 0 (otherwise some representation is included in Σ two or more times). The effectiveness of the method used to obtain a canonical form, also consists in the fact that if all unit cells in the matrix Σ are considered to be one-dimensional, then permutations indecomposable direct terms will be indecomposable and pairwise non-equivalent. It is easy to check also directly. Indeed, the representation Σ is permutatively equivalent to the direct sum of the following representations:

1) $B_1 = (1),\ \ C_1 = (1);$
2) $B_2 = (0),\ \ C_2 = (0);$
3) $B_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix};$
4) $B_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$

each of which is obviously indecomposable and all of them are pairwise non-equivalent.

Hence we have that the representations 1)–4) exhaust all (up to equivalence) indecomposable representations of the semigroup S_d and, therefore, the Auslander algebra can be calculated starting from the representations Σ_0 of the form $(*)$ with one-dimensional unit cells. Therefore, Σ_0 is given by the matrices

$$b \to B_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad c \to C_0 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which means that the Auslander matrix algebra is given by equalities $B_0X = XB_0$, $C_0X = XC_0$ as equations relative to the matrix $X = (x_{ij})$, $1 \le i, j \le 5$.

It is easy to calculate that the equality $B_0X = XB_0$ is equivalent to the equalities $x_{ij} = 0$ for $i = 1, 2, 3, j = 4, 5, 6$; $i = 4, 5, 6, j = 1, 2, 3$; $i = 1, 2, 3, j = 4, 5, 6$; $i = 5, 6, j = 4$; $i = 5, j = 6$ and $x_{44} = x_{55}$ (see, e.g., [17, VIII, §2]).

Now consider the equality $C_0X = XC_0$. We have:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} & 0 & 0 & 0 \\ x_{21} & x_{22} & x_{23} & 0 & 0 & 0 \\ x_{31} & x_{32} & x_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{44} & x_{45} & x_{46} \\ 0 & 0 & 0 & 0 & x_{44} & 0 \\ 0 & 0 & 0 & 0 & x_{65} & x_{66} \end{pmatrix} =$$

$$= \begin{pmatrix} x_{11} & x_{12} & x_{13} & 0 & 0 & 0 \\ x_{21} & x_{22} & x_{23} & 0 & 0 & 0 \\ x_{31} & x_{32} & x_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{44} & x_{45} & x_{46} \\ 0 & 0 & 0 & 0 & x_{44} & 0 \\ 0 & 0 & 0 & 0 & x_{65} & x_{66} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

i.e.

$$\begin{pmatrix} x_{11}+x_{21} & x_{12}+x_{22} & x_{13}+x_{23} & 0 & 0 & 0 \\ x_{21} & x_{22} & x_{23} & 0 & 0 & 0 \\ x_{31} & x_{32} & x_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{11}+x_{12} & x_{13} & 0 & 0 & 0 \\ x_{21} & x_{21}+x_{22} & x_{23} & 0 & 0 & 0 \\ x_{31} & x_{31}+x_{32} & x_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From here we have that

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & 0 & 0 & 0 \\ 0 & x_{11} & 0 & 0 & 0 & 0 \\ 0 & x_{32} & x_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{44} & x_{45} & x_{46} \\ 0 & 0 & 0 & 0 & x_{44} & 0 \\ 0 & 0 & 0 & 0 & x_{65} & x_{66} \end{pmatrix}.$$

Theorem 1 is proved.

4. Conclusions. The paper describes the Auslander matrix algebra over an arbitrary field of characteristic 2 of a unique (up to isomorphism) commutative noncyclic semigroup of the third order without unit and zero elements. Since the Auslander matrix algebra unambiguously defines the category of matrix representations, the obtained result (together with the corresponding research methods) will find application in the study of categories of representations of other semigroups.

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Бондаренко В. М., Зубарук О. В. Про алгебру Аусլендера над полем характеристики два комутативної нециклічної напівгрупи третього порядку без одиничного і нульового елементів.

Класифікація напівгруп третього порядку (в термінах таблиць Келі, з точністю до ізоморфізму та антиізоморфізму) була вперше отримана Т. Тамурою в 1953 році, а пізніше, але вже за допомогою комп'ютерної програми, Г. Е. Форсайтом (в 1955 році). Мінімальні системи твірних і відповідні визначальні співвідношення для всіх таких напівгруп побудовані в працях В. М. Бондаренка та Я. В. Заціхи. Вони також описали зображувальний тип напівгруп третього порядку над довільним полем, а у випадку напівгруп скінченного зображувального типу вказали канонічні форми матричних зображень.

У низці попередніх праць автори досліджували матричні алгебри Аусլендера для напівгруп третього порядку. Ця стаття продовжує такі дослідження.

Ключові слова: напівгрупа, антиізоморфізм, мінімальні системи твірних, визначальні співвідношення, матричне зображення, зображувальний тип, канонічна форма, алгебра Аусլендера.

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