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ON SEMIDISTRIBUTIVE LOCAL NEARRINGS

In [1] it was proved that the additive group of every semidistributive nearring R with an identity is abelian. In this paper we consider finite semidistributive local nearrings. A nearring $R = (R, +, \cdot)$ with identity is said to be local if the set L of all non-invertible elements of R is a subgroup of R^+. It is shown that the semigroup $(L, \cdot)$ of all non-invertible elements of finite semidistributive local nearrings on 2-generated 2-group is commutative.

Keywords: additive group, local nearring, semidistributive local nearring, 2-generated 2-group, semigroup of all non-invertible elements.

1. Preliminaries.

We recall first some basic definitions of the theory of nearrings.

Definition 1. *A set R with two binary operations "$+$" and "$\cdot$" is called a (left) nearring if the following statements hold:*
(1) $(R, +)$ is a (not necessarily abelian) group with neutral element 0;
(2) $(R, \cdot)$ is a semigroup;
(3) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x,\ y,\ z \in R$.

If R is a nearring, then the group $R^+ = (R, +)$ is called the additive group of R. If in addition $0 \cdot x = 0$, then the nearring R is called zero-symmetric and if the semigroup $(R, \cdot)$ is a monoid, i. e. it has an identity element i, then R is a nearring with identity i. In the latter case the group R^* of all invertible elements of the monoid $(R, \cdot)$ is called the multiplicative group of R.

Definition 2 ([1])**.** *A (left) nearring R is called semidistributive if so is the multiplication from the right in respect to its addition. In other words, for any elements $r,\ s,\ t \in R$ the equality $(r + s + r)t = rt + st + rt$ holds.*

It is obvious that every distributive nearring is semidistributive, but not conversely. For example, the nearring $Map(G)$ of all functions on the group G of order 2 is semidistributive and not distributive.

Recall that an element t of a nearring R is called distributive in R if $(r + s)t = rt + st$ for any elements $r,\ s$ of R.

It is well-known that the additive group of any distributive nearring with identity is abelian. The following two assertions were proved in [1].

Lemma 1. *The additive group of every semidistributive nearring R with an identity is abelian.*

Lemma 2. *Let R be a semidistributive nearring with an identity. Then the elements of odd orders of the additive group of R are distributive in R. In particular, each semidistributive nearring of odd order is a ring.*

2. Finite semidistributive local nearrings.

Maxson shown in [6] that every non-cyclic abelian p-group of order $p^n > 4$ is the additive group of a zero-symmetric local nearring which is not a ring.

Definition 3. *A nearring R with identity is said to be local if the set $L = R \setminus R^*$ of all non-invertible elements of R is a subgroup of R^+.*

The following lemma characterizes the main properties of finite local nearrings (see [3]).

Lemma 3. *Let R be a finite local nearring with identity i and L be the subgroup of R^+ of all non-invertible elements from R. Then R^+ is a p-group for a certain prime number p whose exponent is an additive order of the identity i.*

The following result determines the structural feature of finite local nearrings.

Proposition 1. *Each non-trivial subnearring with identity of a finite local nearring is a local nearring.*

Proof. Let $R = (R, +, \cdot)$ be a finite local nearring and L be the subgroup of R^+ of all non-invertible elements from R. Let R_1 be a non-trivial subnearring with identity in R and $(L_1, +)$ be the semigroup of non-invertible elements of R_1. Since $(L_1, +)$ is a subsemigroup of L it follows that $(L_1, +)$ is a subgroup of L, and hence a subgroup in ${R_1}^+$. Hence R_1 is a local nearring by Definition 3. The statement is proved.

As a direct corollary of Lemmas 1, 2 and 3 we have the following statement.

Lemma 4. *Let R be a finite semidistributive local nearring which is not a ring. Then R^+ is an abelian 2-group.*

Let R be a finite semidistributive local nearring on 2-generated 2-group R^+. Hence R^+ is an abelian group of type $(2^m, 2^n)$ with $m \geq n \geq 1$ as a corollary of Lemma 4. Let $|R : L| = 2^k$ with $1 \leq k < m + n$. Then $R^+ = \langle a \rangle + \langle b \rangle$, where $a2^m = b2^n = 0$ with $m \geq n \geq 1$ and $a + b = b + a$. Hence R^+ is of exponent 2^m and, so a coincides with identity of R by Lemma 3. Moreover, each element $x \in R$ is uniquely written in the form $x = ax_1 + bx_2$ with coefficients $0 \leq x_1 < 2^m$ and $0 \leq x_2 < 2^n$. So that $xa = ax = x$ for each $x \in R$. Furthermore, for each $x \in R$ there exist uniquely determined integers $\alpha(x) \in Z_{2^m}$ and $\beta(x) \in Z_{2^n}$ such that $xb = a\alpha(x) + b\beta(x)$ and so some mappings $\alpha : R \to Z_{2^m}$ and $\beta : R \to Z_{2^n}$ are determined. So $b \in L$, whence $L = \langle a2^k \rangle + \langle b \rangle$. Furthermore, $R^* = R \setminus L$ and so an element $x = ax_1 + bx_2$ belongs to R^* if and only if $x_1 \not\equiv 0 \pmod{2^k}$.

Lemma 5. *Let $x = ax_1 + bx_2$ and $y = ay_1 + by_2$ be elements of R. Then*

$$xy = a(x_1y_1 + \alpha(x)y_2) + b(x_2y_1 + \beta(x)y_2).$$

Moreover, for the mappings $\alpha : R \to Z_{2^m}$ and $\beta : R \to Z_{2^n}$ the following statements hold:

(0) $\alpha(0) = \beta(0) = 0$ *if and only if the nearring R is zero-symmetric*;

(1) $\alpha(a) = 0$ *and* $\beta(a) = 1$;
(2) $\alpha(x) \equiv 0 \pmod{2^{m-n}}$;
(3) $\alpha(xy) = x_1\alpha(y) + \alpha(x)\beta(y)$;
(4) $\beta(xy) = x_2\alpha(y) + \beta(x)\beta(y)$.

Proof. As $0 \cdot a = a \cdot 0 = 0$, the nearring R is zero-symmetric if and only if $0 = 0 \cdot b = a\alpha(0) + b\beta(0)$ whence $\alpha(0) = \beta(0) = 0$, proving statement (0). In addition, from the equality $b = ab = a\alpha(a)+b\beta(a)$ it implies $\alpha(a) = 0$ and $\beta(a) = 1$, and so statement (1) holds. Next, by the left distributive law, we have

$$xy = (xa)y_1 + (xb)y_2 = (ax_1 + bx_2)y_1 + (a\alpha(x) + b\beta(x))y_2 =$$

$$= ax_1y_1 + bx_1y_1 + a\alpha(x)y_2 + b\beta(x)y_2 =$$

$$= a(x_1y_1 + \alpha(x)y_2) + b(x_2y_1 + \beta(x)y_2) \tag{$*$}$$

as desired.

Next, by formula (*) for $y = b2^n = 0$ we have $0 = x(b2^n) = a\alpha(x)2^n$. Thus $\alpha(x) \equiv 0 \pmod{2^{m-n}}$, as claimed in (2).

Finally, the associativity of multiplication in R implies that

$$x(yb) = (xy)b = a\alpha(xy) + b\beta(xy).$$

Furthermore, substituting $yb = a\alpha(y) + b\beta(y)$ instead of y in formula (*), we also have

$$xy = a((x_1\alpha(y) + \alpha(x)\beta(y)) + b(x_2\beta(y) + \beta(x)\beta(y)).$$

Comparing the coefficients under a and b in two expressions obtained for $x(yb)$, we derive statements (3) and (4) of the lemma.

Theorem 1. *Let R be a semidistributive local nearring whose additive group R^+ is isomorphic to an abelian group of type $(2^m, 2^n)$ with $m \geq n > 1$. Then the semigroup $(L, \cdot)$ is commutative.*

Proof. If $x = ax_1 + bx_2$ and $y = ay_1 + by_2 \in L$ then $x_1 \equiv 0 \pmod{2^k}$ and $y_1 \equiv 0 \pmod{2^k}$. Let $x_1 = 2s$ and $y_1 = 2t$, where $s,\ t \in N$. Then for each $x,\ y \in L$ using the left distributive and semidistributive laws we have:

$$xy = (ax_1 + bx_2)y = (a2s + bx_2)y = (as + bx_2 + as)y =$$

$$= (as)y + (bx_2)y + (as)y = as(y + y) + (bx_2)y =$$

$$= (as)(y2) + (bx_2)y = as(a2y_1 + b2y_2) + bx_2(ay_1 + by_2) =$$

$$= a2sy_1 + b2sy_2 + bx_2y_1 + a\alpha(b)x_2y_2 + b\beta(b)x_2y_2 =$$

$$= ax_1y_1 + bx_1y_2 + bx_2y_1 + a\alpha(b)x_2y_2 + b\beta(b)x_2y_2 =$$

$$= a(x_1y_1 + \alpha(b)x_2y_2) + b(x_1y_2 + x_2y_1 + \beta(b)x_2y_2).$$

At the same time we get:

$$yx = (ay_1 + by_2)x = (a2t + by_2)x = (at + by_2 + at)x =$$

$$= (at)x + (by_2)x + (at)x = at(x + x) + (by_2)x =$$

$$= (at)(x2) + (by_2)x = at(a2x_1 + b2x_2) + by_2(ax_1 + bx_2) =$$
$$= a2tx_1 + b2tx_2 + by_2x_1 + a\alpha(b)x_2y_2 + b\beta(b)x_2y_2 =$$
$$= ax_1y_1 + bx_2y_1 + bx_1y_2 + a\alpha(b)x_2y_2 + b\beta(b)x_2y_2 =$$
$$= a(x_1y_1 + \alpha(b)x_2y_2) + b(x_1y_2 + x_2y_1 + \beta(b)x_2y_2).$$

Therefore $xy = yx$ for each $x,\ y \in L$ and so $(L, \cdot)$ is commutative, as desired.

As an example, there exist 1068 non-isomorphic local nearrings (LNR) on 2-generated abelian 2-groups of order at most 32, among which 42 are semidistributive (SDLNR). The next table is obtained from the packages SONATA and LocalNR [9] of the computer algebra system GAP.

Additive Group	Number of LNR	Number of SDLNR
$C_2 \oplus C_2$	2	2
$C_4 \oplus C_2$	5	5
$C_4 \oplus C_4$	29	9
$C_8 \oplus C_2$	23	5
$C_8 \oplus C_4$	880	16
$C_{16} \oplus C_2$	129	5

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Раєвська М. Ю. Про напівдистрибутивні локальні майже-кільця.

В [1] було доведено, що адитивна група кожного напівдистрибутивного майже-кільця R з одиницею є абелевою. В цій статті розглядаються скінченні напівдистрибутивні локальні майже-кільця. Майже-кільце $R = (R, +, \cdot)$ з одиницею називається локальним, якщо множина L всіх необоротних елементів з R є підгрупою в R^+. Показано, що напівгрупа $(L, \cdot)$ всіх необоротних елементів скінченного напівдистрибутивного локального майже-кільця на 2-породженій 2-групі є комутативною.

Ключові слова: адитивна група, локальне майже-кільце, напівдистрибутивне локальне майже-кільце, 2-породжена 2-група, напівгрупа всіх необоротних елементів.

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