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AVERAGED OPTIMAL CONTROL PROBLEMS OF NON-LINEAR DIFFERENTIAL INCLUSIONS ON THE FINITE AND INFINITE INTERVALS

In this paper, we use the averaging method to find an approximate solution for the optimal control of nonlinear differential inclusions with rapidly oscillating coefficients.

This work highlights the interplay between averaging methods and asymptotic analysis, suggesting that a hybrid approach can provide reliable strategies for solving complex optimal control problems. Addressing both finite interval and unbounded domains, the studies together contribute to a more complete framework for understanding and applying control methodology in nonlinear settings. We use the averaging method to find an approximate solution to these optimal control problems. Future research can benefit from integrating insights from both methodologies to further improve control strategies, potentially leading to improved results in various fields of engineering and applied mathematics. The Carathéodory-type differential switching optimal control problems are considered.

Keywords: non-linear differential inclusion, optimal control, averaging method, approximate solution.

1. Introduction. In recent years, the study of optimal control problems involving differential inclusions has gained significant attention due to its wide range of applications in engineering, economics, and the natural sciences. This article address the complexities of controlling systems described by differential inclusions, albeit from different perspectives and methodologies.

The first part of research explores the application of the averaging method to non-linear differential inclusions within a finite interval, providing insights into simplifying complex control systems for practical implementation. The second one delves into the asymptotic behavior of optimal control problems on the semiaxes, particularly focusing on Carathéodory differential inclusions with fast oscillating coefficients, thereby offering a deeper understanding of the impact of rapid oscillations on system dynamics. Together, these works contribute to advancing the theoretical framework and application strategies for optimal control in systems governed by differential inclusions.

2. Statement of the problem. Let us consider two optimal control problems. The first one

$$\begin{cases} \dot{x}(t) \in X(\frac{t}{\varepsilon}, x(t), u(t)), t \in (0,T), \\ x(0) = x_0, u(\cdot) \in U, \\ J[x,u] = \int\limits_0^T L(t, x(t), u(t))dt + \Phi(x(T)) \to \inf. \end{cases} \tag{1}$$

Here $\varepsilon > 0$ is a small parameter, $x : [0, T] \to \mathbb{R}$ is an unknown phase variable, $u : [0, T] \to \mathbb{R}^m$ is an unknown control function, $X : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \text{conv}(\mathbb{R}^n)$ is a multi-valued function, $U \subset L^2(0, T)$ is a fixed set.

Assume that uniformly with respect to x for every $u \in \mathbb{R}^m$

$$\text{dist}_H \left(\frac{1}{s} \int_0^s X(\tau, x, u) d\tau, Y(x, u) \right) \to 0, \quad s \to \infty, \tag{2}$$

where limits for multi-valued function are considered in the sense of [1, 2], $dist_H$ is the Hausdorff metric, $Y : \mathbb{R}^n \times \mathbb{R}^m \to \text{conv}(\mathbb{R}^n)$, and the integral of multi-valued function is considered in the sense of Aumann [3]. We consider the following problem with averaged right hand side:

$$\begin{cases} \dot{y}(t) \in Y(y(t), u(t)), \\ y(0) = x_0, u(\cdot) \in U, \\ J[x, u] = \int\limits_0^T L(t, y(t), u(t)) dt + \Phi(x(T)) \to \inf. \end{cases} \tag{3}$$

Under the natural assumptions on X, L, Φ, U we will show that the problems (1) and (3) have solutions $\{\overline{x}_\varepsilon, \overline{u}_\varepsilon\}$ and $\{\overline{y}, \overline{u}\}$ respectively,

$$\overline{J}_{\varepsilon_n} \to \overline{J}, \quad \varepsilon_n \to 0,$$

where $\overline{J}_{\varepsilon_n} := J[\overline{x}_{\varepsilon_n}, \overline{u}_{\varepsilon_n}]$, $\overline{J} := [\overline{y}, \overline{u}]$, and up to a subsequence

$$\overline{u}_{\varepsilon_n} \to \overline{u} \text{ in } L^2(0, T),$$

$$\overline{x}_{\varepsilon_n} \to \overline{y} \text{ in } \mathbb{C}([0, T]).$$

In what follows we consider the problem of finding an approximate solution of (1) by transition to averaged coefficients. We note that the transition to the averaging parameters can essentially simplify the problem.

We consider the second following optimal control problem

$$\begin{aligned} \dot{x}(t) \in f(\frac{t}{\varepsilon}, x(t)) + g(x(t))u(t), \quad t \geq 0, \\ x(0) = x_0, \end{aligned} \tag{4}$$

$$u \in \mathfrak{U} = \{u \in L^2(0, \infty; \mathbb{R}^m) |\ u(t) \in U \text{ a.e. on } (0, \infty)\} \tag{5}$$

is such that

$$J(x, u) = \int_0^\infty (e^{-\gamma t} \varphi(x(t)) + |u(t)|^2) dt \ \to \ \inf, \tag{6}$$

where $\varepsilon > 0$ is a small parameter, and f, g, φ satisfy the following:

1) $f : [0, \infty) \times \mathbb{R}^d \to \text{conv}$;

2) $\forall x \in \mathbb{R}^d$ the map $f(\cdot, x)$ possesses a measurable selector;

3) $\forall t \geq 0$ the map $f(t, \cdot)$ is upper semicontinuous;

4) $\exists M \geq 0\ \forall x \in \mathbb{R}^d\ \forall t \geq 0 : \quad ||f(t, x)||_+ \leq M$;

5) $g : \mathbb{R}^d \to \mathbb{R}^{d \times m}$ is continuous and bounded, that is $\exists N \geq 0$ with $||g(x)|| \leq N, \quad x \in \mathbb{R}^d$;

6) $U \subset \mathbb{R}^m$ is closed, convex and $0 \in U$;
7) $\varphi : \mathbb{R}^d \to \mathbb{R}$ is continuous and there are constants $c > 0$ and $p \geq 1$ with

$$\inf_{x \in \mathbb{R}^d} \varphi(x) \geq -c, \quad |\varphi(x)| \leq c(1 + ||x||^p).$$

For a given control function $u \in \mathcal{U}$ we understand solution of (1) as an absolutely continuous function x which satisfies (1) almost everywhere (a.e.) on $[0, +\infty)$. In this case we say that $\{x, u\}$ is an admissible pair for (1)-(3). An admissible pair $\{x^\varepsilon, u^\varepsilon\}$ is called an optimal pair (or solution) for (1)-(3) if for every admissible pair $\{x, u\}$ we have

$$J(x^\varepsilon, u^\varepsilon) \leq J(x, u).$$

The existence of an optimal solution $\{x^\varepsilon, u^\varepsilon\}$ is established in the next section.

Let us denote

$$J^\varepsilon := \inf J(x, u) = J(x^\varepsilon, u^\varepsilon).$$

Using approach of [5] we define the average function $\bar{f}$ basing on the notion of the Kuratowski upper limit [6]

$$\bar{f}(x) = \cap_{\delta > 0} \bar{F}^\delta(x),$$

where $\bar{F}^\delta$ is the convex hull of the map

$$\Phi^\delta(x) = \limsup_{\theta \nearrow 1} \limsup_{T \to \infty} \frac{1}{(1-\theta)T} I(\theta T, T, x, \delta),$$

$$I(\theta T, T, x, \delta) = \Big\{ \int_{\theta T}^{T} v(t)dt \mid v(\cdot) \in L^1_{loc}(0, \infty; \mathbb{R}^d), \quad v(t) \in f(t, y), \ y \in \overline{O_\delta(x)} \Big\}.$$

It is proved in [5] that if there exists $\bar{F}(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t, x)dt$ in the sense of the Hausdorff distance $dist_H$, and if $f(t.\cdot)$ is Lipschitz, then $\bar{f} = \bar{F}$.

Also we consider the optimal control problem

$$\dot{x} \in \bar{f}(x) + g(x)u(t), \quad x(0) = x_0, \tag{7}$$

$$u \in \mathcal{U}, \tag{8}$$

$$J(x, u) \to \inf \tag{9}$$

Our aim is to prove that for $\varepsilon \to 0$ it follows that

$$J^\varepsilon \to \bar{J} \text{ and } \{x^\varepsilon, u^\varepsilon\} \to \{\bar{x}, \bar{u}\} \text{ in some sense,}$$

where $\{\bar{x}, \bar{u}\}$ is a solution of (7)–(9), $\bar{J} = J(\bar{x}, \bar{u})$.

3. Main resuts. Let $Q = \{t \geq 0, x \in \mathbb{R}^n, u \in \mathbb{R}^m\}$ and assume the following assumptions hold.

(A1) the mapping $t, x, u \mapsto X(t, x, u)$ is continuous in Hausdorff metric;

(A2) the multi-valued function $X(t, x, u)$ satisfies the next growth property: there exists $M > 0$ such that

$$||X(t, x, u)||_+ \leq M(1 + ||x||) \ \forall (t, x, u) \in Q,$$

where $||X(t, x, u)||_+ = \sup_{\xi \in X(t,x,u)} ||\xi||$, $||\xi||$ is the Euclidian norm of $\xi \in \mathbb{R}^n$;

(A3) the multi-valued function $X(t,x,u)$ satisfies the next Lipschitz condition: there exists $\lambda > 0$ such that

$$\mathrm{dist}_H\left(X(t,x_1,u_1), X(t,x_2,u_2)\right) \leq \lambda\left(||x_1-x_2|| + ||u_1-u_2||\right);$$

(A4) the function $(x,u) \mapsto L(t,x,u)$ is continuous, moreover the function $t \mapsto L(t,x,u)$ is measurable $\forall x \in \mathbb{R}^n, u \in \mathbb{R}^m$ and

$$|L(t,x,u)| \leq c(t)(1+||u||),$$

where $c \in L^2(0,T)$ is a given function;

(A5) the function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is continuous;

(A6) the set $U \subset L^2(0,T)$ is compact.

Theorem 1. *Under the Assumptions A1-A6 the problem* (1) *(resp. the problem* (3)*) has the solution* $\{\overline{x}_\varepsilon, \overline{u}_\varepsilon\}$ *(resp.* $\{\overline{y}, \overline{u}\}$*).*

Proof. Fix $\varepsilon > 0$ and suppress it in what follows. Under the conditions on L and Φ the cost functional in (1) reaches its finite extremum. Now deduce a priory estimate for $x(t)$. Since x is an absolutely continuous function then $t \mapsto ||x(t)||$ is an absolutely continuous too and

$$\frac{d}{dt}||x(t)|| \leq ||\dot{x}(t)|| \text{ a.e.}$$

Then

$$\frac{d}{dt}||x(t)|| \leq ||\dot{x}(t)|| \leq ||X(t,x,u)||_+ \leq M(1+||x||),$$

and

$$||x(t)|| \leq ||x(0)|| + \int\limits_0^t M(1+||x||)ds = ||x(0)|| + MT + \int\limits_0^t M||x||ds.$$

Taking into account Gronwall's inequality we have

$$||x(t)|| \leq \left(||x(0)|| + MT\right)e^{\int\limits_0^t Mds} = \left(||x(0)|| + MT\right)e^{Mt} \leq \left(||x(0)|| + MT\right)e^{MT}. \tag{10}$$

Let $\{x_n, u_n\}_{n\in\mathbb{N}}$ be a minimizing sequence for the problem (1), that is:

$$\{x_n, u_n\} \in \{(x,u) : u \subset U, x \text{ is the solution to the Cauchy problem (1) for all admissible } u\},$$

and $J(x_n, u_n,) \leq \overline{J} + \frac{1}{n}$. Due to (10) we have the uniform boundedness of the sequence $\{x_n\}_{n\in\mathbb{N}}$ on every finite interval $[0,T]$, i.e. $\exists L > 0$:

$$\sup_{t\in[0,T]} ||x_n(t)|| \leq L, t \in [0,T].$$

Moveover,

$$\sup_{t\in[0,T]} ||\dot{x}_n(t)|| \leq L, t \in [0,T], \tag{11}$$

and

$$||x_n(t_2) - x_n(t_1)|| \le \int_{t_1}^{t_2} M(1+L)ds = M(1+L)(t_2 - t_1),$$

so the sequence $\{x_n\}_{n\in\mathbb{N}}$ is precompact in $\mathbb{C}([0,T])$. Due to the Arcel's theorem $x_n \to \overline{x}$ in $C([0,T])$ up to a subsequence.

From [2] and (11) we deduce that $\overline{x}$ is absolutely continuous and $\dot{x}_n \to \dot{\overline{x}}$ $*$-weakly as $n \to \infty$ in $L^\infty(0,T)$. Since $\forall \varepsilon > 0$ for a.e. t there exists n_0 such that $\forall n \ge n_0$

$$\lambda\left(||x_n(t) - \overline{x}(t)|| + ||u_n(t) - \overline{u}(t)||\right) < \varepsilon,$$

then by the Assumption A3 we have

$$\dot{x}_n(t) \in X\left(\frac{t}{\varepsilon}, x_n(t), u_n(t)\right) \subset O_\varepsilon\left(X\left(\frac{t}{\varepsilon}, x_n(t), u_n(t)\right)\right).$$

Taking into account the convergence theorem [4, p.60] for a.e. t we have

$$\dot{\overline{x}}(t) \in X\left(\frac{t}{\varepsilon}, \overline{x}(t), \overline{u}(t)\right).$$

By the Assumption A6 we obtain the convergence $u_n \to \overline{u}, n \to \infty$ in $L^2[0,T]$ up to a subsequence.

Now we will show that $\{\overline{x}, \overline{u}\}$ is the solution of (1). Since $u_n(t) \to \dot{\overline{u}}(t)$ and $x_n(t) \to \overline{x}(t), n \to \infty$ a.e. by the Assumption A4 we obtain that

$$L(t, x_n(t), u_n(t)) \to L(t, \overline{x}(t), \overline{u}(t)) \text{ a.e.}, \quad n \to \infty,$$

and $\{L(t, x_n, u_n)\}$ is bounded in $L^2(0,T)$. Therefore by the Lions' Lemma we have that $L(t, x_n, u_n) \to L(t, \overline{x}, \overline{u})$ weakly in $L^2(0,T)$ for $n \to \infty$.
Hence,

$$\int_0^T L(t, x_n(t), u_n(t))dt \to \int_0^T L(t, \overline{x}(t), \overline{u}(t))dt, \quad n \to \infty,$$

With the convergence $\Phi(x_n(T)) \to \Phi(\overline{x}(T))$ we have $\lim\limits_{n\to\infty} J[x_n, u_n] = J[\overline{x}, \overline{u}] = \overline{J}$ and therefore $\{\overline{x}, \overline{u}\}$ is the solution of (1).

For the second optimal control problem we have a similar result.

Lemma 1. *Under the conditions 1)–7) the optimal control problem 1–3 has a solution* $\{x^\varepsilon, u^\varepsilon\}$.

Lemma 2. *Let* $f : [0,\infty) \times \mathbb{R}^d \to conv$ *satisfy 1)–4). Then there exists a sequence of locally Lipschitz maps* $f^k : [0,\infty) \times \mathbb{R}^d \to conv$ *satisfying 1)–4) for* $k \in \mathbb{N}$ *with*

$$f(t,x) \subset \cdots \subset f^{k+1}(t,x) \subset f^k(t,x), \quad t \ge 0,\ x \in \mathbb{R}^d,\ k \in \mathbb{N} \tag{12}$$

and for each $k \in \mathbb{N}$ *and* $x \in \mathbb{R}^d$ *there exist* $l_k > 0$ *and* $\delta_k > 0$ *such that*

$$dist_H(f^k(t,x'), f^k(t,x'')) \le l_k\|x' - x''\|, \quad x', x'' \in O_{\delta_k}(x),\ t \ge 0, \tag{13}$$

moreover, for any $\varepsilon > 0,\ t \ge 0,\ x \in \mathbb{R}^d$ *there is* $K = K(\varepsilon, t, x)$ *with*

$$f^k(t,x) \subset \overline{co} f(t, O_\varepsilon(x)), \quad k \ge K. \tag{14}$$

Moreover for a fixed $\varepsilon > 0$, if x_n is the solution to (4) corresponding to the control u_n, $n \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} ||u_n||_{L^2} < \infty$ then up to a subsequence the following convergence holds

$$u_n \to u \text{ weakly in } L^2(0,\infty;\mathbb{R}^m), \tag{15}$$

$$x_n \to x \text{ in } C([0,T];\mathbb{R}^d), \quad T > 0, \tag{16}$$

where x is a solution of (4) with control u. Additionally, if $u_n \in \mathfrak{U}$ for $n \in \mathbb{N}$ then $u \in \mathfrak{U}$.

Lemma 3. *Let $\varepsilon_n \to 0$ as $n \to \infty$ and x_n be a solution of (4) with control u_n. Let $\{x_n, u_n\} \to \{x, u\}$ as $n \to \infty$ in the sense of (15),(16). Then x is a solution of (7) with control u.*

Now we are in a position to prove our main result.

Theorem 2. *Assume that conditions 1)–7) are satisfied. Assume that for any $u \in \mathfrak{U}$ the problem (7) has a unique solution. Let $\{x^\varepsilon, u^\varepsilon\}$ be an optimal pair in (4)–(6), $J^\varepsilon = J(x^\varepsilon, u^\varepsilon)$. Then*

$$J^\varepsilon \to \bar{J} \quad \textit{for} \quad \varepsilon \to 0, \tag{17}$$

and for $\varepsilon_n \to 0$ it holds that

$$x^{\varepsilon_n} \to \bar{x} \quad \textit{in} \quad C([0,T];\mathbb{R}^d), \quad T > 0 \tag{18}$$

$$u^{\varepsilon_n} \to \bar{u} \quad \textit{weakly in} \quad L^2(0,\infty;\mathbb{R}^m), \tag{19}$$

where $\{\bar{x}, \bar{u}\}$ is an optimal pair in (7)–(9), $\bar{J} = J(\bar{x}, \bar{u})$.

Proof.
Let for $\varepsilon_n \to 0$, $\{x^{\varepsilon_n}, u^{\varepsilon_n}\}$ be an optimal pair for (4)-(6). From the optimality of u_n^ε it follows that

$$J(x^{\varepsilon_n}, u_n^\varepsilon) \le J(x_n, 0),$$

where x_n is a solution of (4) with $\varepsilon = \varepsilon_n$, $u = 0$. Then

$$-\frac{c}{\gamma} + \|u_n^\varepsilon\|^2 \le \int_0^\infty e^{-\gamma t}\varphi(x_n(t)) \le \int_0^\infty e^{-\gamma t} c(1 + (\|x_0\| + (M+N)t)^p)dt \le C_1, \tag{20}$$

where C_1 does not depend on n. Additionally, we have

$$\|x_n^\varepsilon(t) - x_n^\varepsilon(s)\| \le M|t-s| + N|t-s|^{\frac{1}{2}}\|u_n^\varepsilon\|. \tag{21}$$

Estimations (20),(21) and the Arzela-Ascoli theorem imply that on some subsequence $\{x^{\varepsilon_n}, u^{\varepsilon_n}\}$, $n \in \mathbb{N}$ converges to some $\{\bar{x}, \bar{u}\}$ in the sense of (18),(19). Hence, from Lemma 3 we deduce that $\bar{x}$ is a solution of (7) with control $u \in \mathfrak{U}$. Let us prove that $\{\bar{x}, \bar{u}\}$ is an optimal pair.

For every $u \in \mathfrak{U}$ and the corresponding solution x_n to (4) we have

$$J(x^{\varepsilon_n}, u^{\varepsilon_n}) \le J(x_n, u). \tag{22}$$

Arguing as in the proof of Lemma 1, we get from (22) after passing to the limit:

$$J(\bar{x}, \bar{u}) \le \lim \inf_{n\to\infty} J(x^{\varepsilon_n}, u^{\varepsilon_n}) \le \lim \inf_{n\to\infty} J(x_n, u). \tag{23}$$

Due to (21) with u_n^ε replaced with u we have that $x_n \to x$ in the sense of (18). By Lemma 3 it follows that x is a unique solution of (7) with control u. So, from (23) follows

$$J(\bar{x}, \bar{u}) \leq \lim \inf_{n\to\infty} J(x_n, u) = J(x, u).$$

This inequality means that $\{\bar{x}, \bar{u}\}$ is an optimal pair.

Applying previous arguments with $u = \bar{u}$ we get

$$J(\bar{x}, \bar{u}) \leq \lim \inf_{n\to\infty} J^{\varepsilon_n} \leq \lim \sup_{n\to\infty} J^{\varepsilon_n} \leq \lim_{n\to\infty} J(x_n, \bar{u}) = J(\bar{x}, \bar{u}).$$

This means, that there exists $\lim_{n\to\infty} J^{\varepsilon_n} = J(\bar{x}, \bar{u})$. Because of arbitrariness of $\varepsilon_n \to 0$, we get (17). Theorem is proved.

4. Concusions. We significantly advance our understanding of optimal control problems characterized by non-linear dynamics and varying conditions. The first study introduces the averaging method as a viable technique to address finite interval problems, elucidating how it simplifies the control process by reducing the complexity of the differential inclusions involved. This approach not only facilitates the identification of optimal solutions but also broadens the applicability of control strategies to a wider range of practical scenarios.

Conversely, the second article delves into the asymptotic behavior of control problems defined on semiaxes, where the challenges posed by fast oscillating coefficients are meticulously examined. The authors demonstrate that under specific conditions, asymptotic techniques can effectively yield optimal control solutions, thereby enhancing our ability to manage systems with rapidly changing dynamics.

Together, these works highlight the interplay between averaging methods and asymptotic analysis, suggesting that a hybrid approach may yield robust strategies for solving complex optimal control problems. By addressing both finite intervals and unbounded domains, the studies collectively contribute to a more comprehensive framework for understanding and applying control methodologies in non-linear settings. Future research could benefit from integrating insights from both methodologies to further refine control strategies, potentially leading to improved outcomes in various engineering and applied mathematics fields.

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Жук Т. Ю. Усереднені задачі оптимального керування нелінійними диференціальними включеннями на скінченних і нескінченних інтервалах.

У даній статті ми використовуємо метод усереднення, щоб знайти наближений розв'язок для оптимального керування нелінійними диференціальними включеннями з швидкоосцилюючими коефіцієнтами.

Дана робота висвітлює взаємодію між методами усереднення та асимптотичним аналізом, припускаючи, що гібридний підхід може дати надійні стратегії для вирішення складних проблем оптимального керування. Звертаючись як до скінченних інтервалів, так і до необмежених областей, дослідження разом роблять внесок у більш повну структуру для розуміння та застосування методології контролю в нелінійних умовах. Ми використовуємо метод усереднення, щоб знайти наближене рішення для цих задач оптимального керування. Майбутні дослідження можуть отримати вигоду від інтеграції розуміння з обох методологій для подальшого вдосконалення стратегій контролю, що потенційно призведе до покращення результатів у різних галузях інженерії та прикладної математики. Розглядаються задачі оптимального керування диференціальним включенням типу Каратеодорі.

Ключові слова: нелінійне диференціальне включення, оптимальне керування, метод усереднення, наближений розв'язок.

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