UDC 512.56
DOI https://doi.org/10.24144/2616-7700.2024.45(2).46-55

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ON MINIMAL MINIMAX SYSTEMS OF GENERATORS FOR POSITIVE POSETS

The representations of partially ordered sets (abbreviated: posets), introduced by L. A. Nazarova and A. V. Roiter (in matrix form) in 1972, play an important role in the modern representation theory and its applications. Yu. A. Drozd proved in 1974 that a poset S has finite representation type over a field if and only if its Tits quadratic form

$$q_S(z) =: z_0^2 + \sum_{i\in S} z_i^2 + \sum_{i<j, i,j\in S} z_i z_j - z_0 \sum_{i\in S} z_i,$$

is weakly positive (i.e., positive on the set of non-negative vectors), but this statement is not true, unlike the case of the quivers, when considering the positive quadratic forms. Hence the posets with positive Tits quadratic forms aroused great interest from various points of view as analogs of Dynkin diagrams. In 2005 the authors described up to isomorphism all posets with positive Tits quadratic form. The basic method for solving this problem is so-called minimax equivalence method proposed by the first author. Recently he introduced some concept (namely, minimax system of generations) with consideration of the corresponding examples, which can be considered as the emergence of a new theory, which study combinatorial properties of posets with respect to minimax equivalence.

In this paper we study from such new point of view the posets with positive Tits quadratic form (which are called positive posets).

Keywords: positive quadratic form, Tits quadratic form, positive poset, minimax equivalence and isomorphism, minimax system of generators.

1. Introduction. When studying the representations of quivers, P. Gabriel [1] introduced a quadratic form of a (finite) quiver Q. This form was called the *Tits quadratic form of the quiver* Q. P. Gabriel proved that the quiver Q is of finite representation type over a field k if and only if its Tits quadratic form is positive. This Gabriel's result laid the foundations of a new direction in the representation theory dealing with the investigation of the relationships between the properties of representations of various objects and the properties of quadratic forms associated with these objects.

In [2], Yu. A. Drozd showed that a (finite) poset S is of finite representation type if and only if its Tits quadratic form

$$q_S(z) = z_0^2 + \sum_{i\in S} z_i^2 + \sum_{i<j, i,j\in S} z_i z_j - z_0 \sum_{i\in S} z_i,$$

is weakly positive, i.e. positive on the non-zero vectors with non-negative coordinates (representations of posets were introduced by L. A. Nazarova and A. V. Roiter in [3]). In contrast to the quivers, the posets with weakly positive and with positive Tits forms do not coincide. Since the connected quivers having positive Tits quadratic form coincide with the quivers whose underlying graphs are (simply faced) Dynkin diagrams [1], the posets with positive Tits form are analogs of the Dynkin diagrams. Therefore investigation related to posets with positive Tits form are important. In [4], [5] the authors classified the posets with positive Tits quadratic form and the minimal posets with non-positive Tits form.

In solving the specified problems, a method based on the notion of minimax equivalence of posets was used (see [6]). In [7] the first author introduced some concept (namely, minimax system of generations) with consideration of the corresponding examples, which can be considered as the emergence of a new theory, which study combinatorial properties of posets with respect to minimax equivalence.

In this paper we study from the new point of view the posets with positive Tits quadratic form. Such posets are called *positive*.

The first author is supported by a grant from the Simons Foundation (1030291, 1290607, V.M.B.).

2. Preliminaries.

2.1. Definitions on posets. Throughout the paper, all posets are finite of order $n > 0$ without an element 0. A poset T is called *dual* to a poset S and is denoted by $S^{\rm op}$ if $T = S$ as usual sets and $x < y$ in T if and only if $x > y$ in S. If S and $S^{\rm op}$ are isomorphic, the poset S is called *self-dual*. T and S are called *anti-isomorphic* if T and $S^{\rm op}$ are isomorphic.

By a subposet we always mean a full one, and singletons are identified with the elements themselves. Sometime (in definitions or statements) we admit empty posets which are or may be later subposets of some posets.

A poset S is called a *sum of subposets* $A_1, A_2, \ldots, A_m$ and write $S = A_1 + A_2 + \cdots + A_m$, if $S = \cup_{i \in S} A_i$ and $A_i \cap A_j = \varnothing$ for each i and $j \neq i$. If any two elements of different summands are incomparable, the sum is called *direct* and is denoted in this case also by $\coprod$ instead of $+$.

A sum $S = A + B$ with $A, B \neq \varnothing$ is said to be *left* (resp. *right*) if $a < b$ (resp. $b < a$) for some $a \in A, b \in B$ and there is no $a' \in A, b' \in B$ such that $a' > b'$ (resp. $b' > a'$). Both left and right sums are called *one-sided*. A sum $S = A + B$ is called *two-sided* if $a < b$ for some $a \in A, b \in B$ and $b' < a'$ for some $a' \in A, b' \in B$. Finally, a one-sided (left or right) or two-sided sum $S = A + B$ is called *minimax* if $x < y$ with x and y belonging to different summands implies that x is minimal and y maximal in S.

2.2. Positive posets. Let S be a poset. *The Tits quadratic form of* S is by definition the following quadratic form $q : \mathbb{Z}^{|S|+1} \to \mathbb{Z}$:

$$q = q_S(z) = z_0^2 + \sum_{i \in S} z_i^2 + \sum_{i<j, i,j \in S} z_i z_j - z_0 \sum_{i \in S} z_i,$$

(see [2]). Here $\mathbb{Z}$ denotes the set of integer numbers and $\mathbb{Z}^{|S|+1}$ consists of the integer vectors (z_i) with $i \in 0 \cup S$. The poset S is called *positive* if so is its Tits quadratic form.

The positive posets were first described in [4] (for posets of width two) and [5] (in general case). They can be of two types: serial and non-serial.

A positive poset S is called *serial* if for any $m \in \mathbb{N}$, there is a positive poset $S(m) \supset S$ such that $|S(m) \setminus S| = m$, and *non-serial* otherwise. There are 108 non-serial positive posets up to isomorphism and duality [5, Table 1] (see also below Section 5).

Now formulate two theorems on positive serial posets (see [4], [5] and also [7]).

A linear ordered set with $n \geq 0$ elements is called a *chain of length* n. A poset of the foem $a_1 < \ldots < a_p < \{b, c\} < d_1 < \ldots d_q\, (p, q \geq 0)$ with one pair of incomparable elements is called an *almost chain of length* $n = p + q + 1$.

Theorem 1. *A poset T is serial positive if and only if it is isomorphic to one of the following poset S:*

(1) *S is a direct sum of a chain of length $k \geq 0$ and a chain of length $s \geq 1$, where $k \leq s$;*

(2) *S is a left minimax sum of two chains of lengths $k \geq 1$ and $s \geq 1$, where $k + s > 3$;*

(3) *S is a direct sum of a chain of length $k \geq 0$ and an almost chain of length $s \geq 1$, where $k + s > 1$.*

Moreover, all these posets are pairwise non-isomorphic.

Theorem 2. *Any positive poset of order $n < 5$ or $n > 7$ is serial.*

3. Main results.

3.1. Minimax equivalence of posets. This concept was introduced by the first author in [6] and studied in detail in [5].

For a poset S and its minimal (resp. maximal) element a, let us $T = S_a^{\uparrow}$ (resp. $T = S_a^{\downarrow}$) denotes the following poset: $T = S$ as usual sets, $T \setminus a = S \setminus a$ as posets, the element a is maximal (resp. minimal) in T, and a is comparable with x in T if and only if they are incomparable in S. Two posets S and T are called (min, max)-*equivalent* if there are posets $S_1, \ldots, S_p$ $(p \geq 0)$ such that, if we put $S = S_0$ and $T = S_{p+1}$, then, for every $i = 0, 1, \ldots, p$, either $S_{i+1} = (S_i)_{x_i}^{\uparrow}$ or $S_{i+1} = (S_i)_{y_i}^{\downarrow}$. Obviously, any poset is (min, max)-equivalent to itself (if one put $p = 0$). Since some time we also use the term *minimax equivalence*.

The notion of minimax equivalence can be naturally continued to the notion of *minimax isomorphism*: posets S and S' are minimax isomorphic if there exists a poset T which is minimax equivalent to S and isomorphic to S'.

In the case when for every $i = 0, 1, \ldots, p$‘ one has $S_{i+1} = (S_i)_{x_i}^{\uparrow}$, T and S are called min-*equivalent*.

Proposition 1. *The following conditions are equivalent:*

(1) *T and S are* (min, max)*-equivalent;*

(2) *T and S are* min*-equivalent.*

In a similar way one can defines the notion of max-equivalence.

3.2. Minimax systems of generators. The concept of such systems of generators was introduced by the first author in [7].

Let $\mathcal{K}$ be a class of finite posets closed under isomorphism and duality (or, equivalently, isomorphism and anti-isomorphism), and let $U = \{U_i\}$ be a set of posets $U_i \in \mathcal{K}$ with i running through a (finite or infinite) set I. The set U is called a *minimax system of generators* of $\mathcal{K}$ if any $X \in \mathcal{K}$ is minimax isomorphic to a poset U_i for some $i \in I$. In the case when any proper subset of U is not a minimax system of generators, the system of generators U is called *minimal.* The system U is called *self-dualI* is so are all U_i.

Example. From [5, Theorem 2] it follows that the minimal posets with non-positive Tits quadratic form (which are called P-critical) has a self-dual minimal minimax system from 5 generators, consisting of the Kleiner posets [8].

3.3. Formulations of the maim theorems. Recall that the Hasse diagram of a poset S is a type of diagram that represents S in the plane. Namely, for a poset S one represents each element of S as a vertex and each pair of elements x, y of S, such that y covers x (i. e. $x < y$ and there is no z satisfying $x < z < y$), as an edge (a line segment or curve) that goes upward from x to y. We call a poset S *quasi-chained* if $H(S)$ is a chain.

Theorem 3. *The classes of serial and non-serial positive posets have minimax systems of quasi-chained generators. Moreover, one can assume that each generator is is of width 2 or self-dual.*

Theorem 4. *The class of serial positive posets of even order has a self-dual minimax system of quasi-chained generators, but the class of those of odd order does not have.*

Theorem 5. *The class of non-serial positive posets does not have a self-dual minimax system of quasi-chained generators.*

Theorem 6. *The class of non-serial positive posets has a minimax system of quasi-chained generators of width* 3, *but the class of serial ones does not have.*

Under proving these theorems, in each case when (by the formulation of some theorem) the indicated system exists we even indicate such a minimal system.

4. Proofs of the theorems.

4.1. The case of serial posets. Let S be a serial poset (see Theorem 1). From the definition of $S_x^{\uparrow}$ we have the following statements:

(4.1.1) if S is of the form (1), $S := S_{1ks} = \{a = a_1 < \ldots < a_k\} \coprod \{b = b_1 < \ldots b_s\}$, then (when $k \neq 0$) $S_a^{\uparrow} \cong S_{1,k-1,s+1}$ and $S_b^{\uparrow} \cong S_{1,k+1,s-1}$;

(4.1.2) if S is of the form (2), $S := S_{2ks} = \{a = a_1 < \ldots < a_k\} \coprod \{b = b_1 < \ldots b_s\}$ with $a_1 < b_s$, then $S_a^{\uparrow}$ is isomorphic to the poset $\{a_2 < \ldots < a_k\} \coprod \{b_1 < \ldots b_{s-1} < (b_s, a_1)\}$ of the form (3) (or to $S(3, k-1, s, s-1)$ in the notation of (4.1.3)) and (when $s > 1$) $S_b^{\uparrow} \cong S_{2,k+1,s-1}$;

(4.1.3) if S is of the form (3), $S := S_{3kst} = \{a = a_1 < \ldots < a_k\} \coprod \{b = b_1 < \ldots b_t < (c, d) < b_{t+1} < \ldots < b_{s-1}\}$, then (when $k \neq 0$) $S_a^{\uparrow} \cong S_{3,k-1,s+1,t}$, $S_b^{\uparrow} \cong S_{3,k+1,s-1,t-1}$ when $t \neq 0$ and $S_c^{\uparrow} \cong S_{2,s,k+1}$ when $t = 0$.

A simple analysis shows that these statements (with using the min-equivalence by Proposition 1) emply that

(4.1.4) the posets $S_{1,0,n}$ with n running through $\mathbb{N}$ and $S_{2,1,m}$ with m running through $\mathbb{N} \setminus 1$ form a minimal minimax system of quasi-chained generators of the class of serial positive posets;

(4.1.5) the posets $S_{1,0,2n}$ with n running through $\mathbb{N}$ and $S_{2,n,n}$ with n running through $\mathbb{N} \setminus 1$ form a minimal self-dual minimax system of quasi-chained generators of the class of serial positive posets of even order;

(4.1.6) for any fixed $n > 0$, the poset $S_{2,1,2n+1}$ is not minimax isomorphic to a self-dual quasi-chained posets.

(4.1.7) for any fixed $m > 0$, the poset $S_{1,0,m}$ is not minimax isomorphic to a poset of width 3.

Obviously, the first part of Theorem 3 (for serial posets) follows from (4.1.4), Theorem 4 follows from (4.1.5) and (4.1.6), the second part of Theorem 5 follows from (4.1.6), and the second part of Theorem 6 follows from (4.1.7).

4.2. The case of non-serial posets. Consider now the case of non-serial posets using the language of Hasse diagrams. Such posets were classified by the authors in [5]; see below the corresponding table in Section 5. We will refer to this table in following the following discussion, calling it Main table.

Proposition 2. *All posets of Main table are divided on 8 classes with respect to minimax isomorphism:*

(I) 1, 2, 3, $\overline{4}$, 46, 47, $\overline{49}$;

(II) 5, 48, $\overline{50}$;

(III) 6, 7, 8, 9, 10, 11, $\overline{12}$, 13, 51, 52, 53, 54, 57, 60, $\overline{61}$;

(IV) 14, 15, 16, 17, 18, 19, 20, 55, 56, 58, 59, $\overline{62}$, $\overline{63}$, 64, $\overline{65}$, $\overline{66}$, 67;

(V) 21, 22, 23, 24, 25, 26, 27, $\overline{28}$, 29, 30, 68, 69, 70, 71, 76, 77, 79, 87, $\overline{88}$, 89;

(VI) 31, 32, 36, 37, 40, 41, 43, 72, 74, 80, 81, 82, 86, $\overline{90}$, $\overline{92}$, 94, 96, $\overline{98}$, 99, 103, 108;

(VII) 33. 34, 35, 38, 39, 42, 44, 73, 75, 78, 83, 84, 91, 93, $\overline{95}$, 97, $\overline{100}$, $\overline{102}$, 104, 106, 107;

(VIII) 45, 85, $\overline{101}$, 105.

The upper underlined posets (and only they) are quisi-chained.

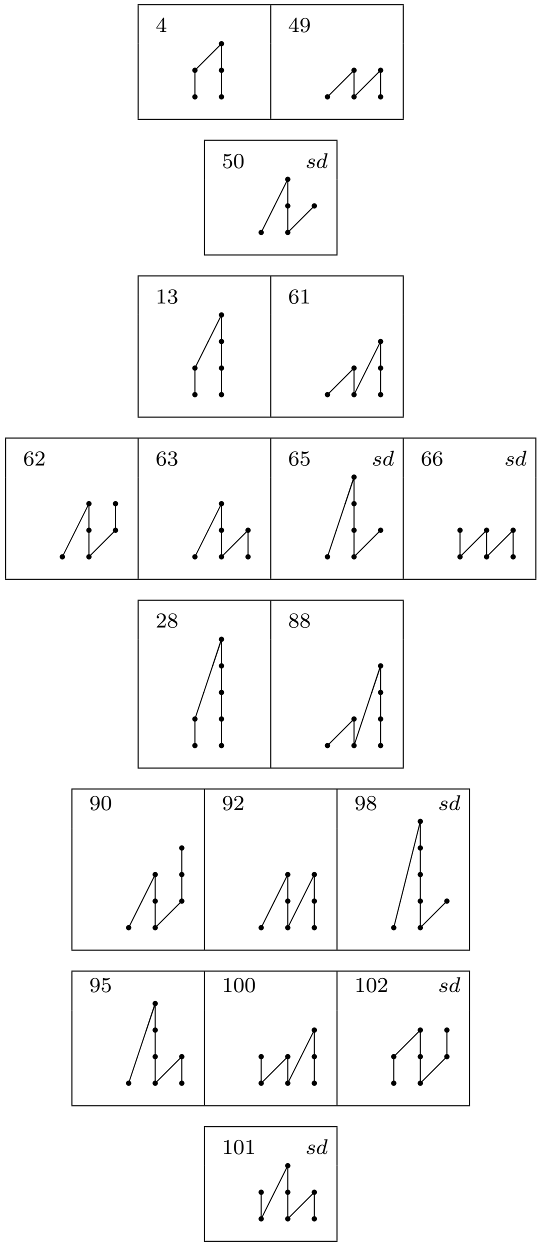
This statement follows from the result of the paper in [5], but some explanations are required.

If one uses the terminology of Subsection 3.2, in [5] the authors show that the set of posets of width two $M = \{1, 5, 6, 14, 21, 31, 33, 45\}$ together with the set M^{op} (consisting of their duals) form a minimal minimax system of generators for all non-serial positive posets. Moreover, from the proofs in [5] it follows that each poset of Main table and its dual are minimax isomorphic to the same poset of M. In other words, each class minimax isomorphism is closed under duality. Therefore, the set M itself is a minimal minimax system of generators for the non-serial positive posets.

From all that has been said it follows that if one writes out all non-serial positive posets not only with accuracy up to isomorphism, but also with accuracy to duality (Main table is exactly this), then it is correct to consider classes of minimax isomorphism with accuracy to duality too.

But in the case when the classes are considered up to duality and we want to make sure that some property is not satisfied for posets of some class, then this property must be closed under duality. All property of non-serial positive posets that will be considered below will satisfy this requirement by silence.

It is easy to see that the sets $QC(\mathrm{I}) \ldots$ QC(VIII) of quasi-chained posets of the classes (I) ... (VIII) consist of, respectively, the following ones (the symbol *sd* denotes self-dual posets):



From the description of the sets $QC(\mathrm{I}), \ldots,$ QC(VIII) the next statements follow:

(4.2.1) for $N = \mathrm{I}, \mathrm{IV}, \mathrm{VI}$, the sets $QC(\mathrm{N})$ contain quasi-chained posets of width 2, and for the rest N contain self-dual quasi-chained posets;

(4.2.2) there is not such N that the set $QC(\mathrm{N})$ contains both a quasi-chained poset of width 2 and a self-dual quasi-chained poset;

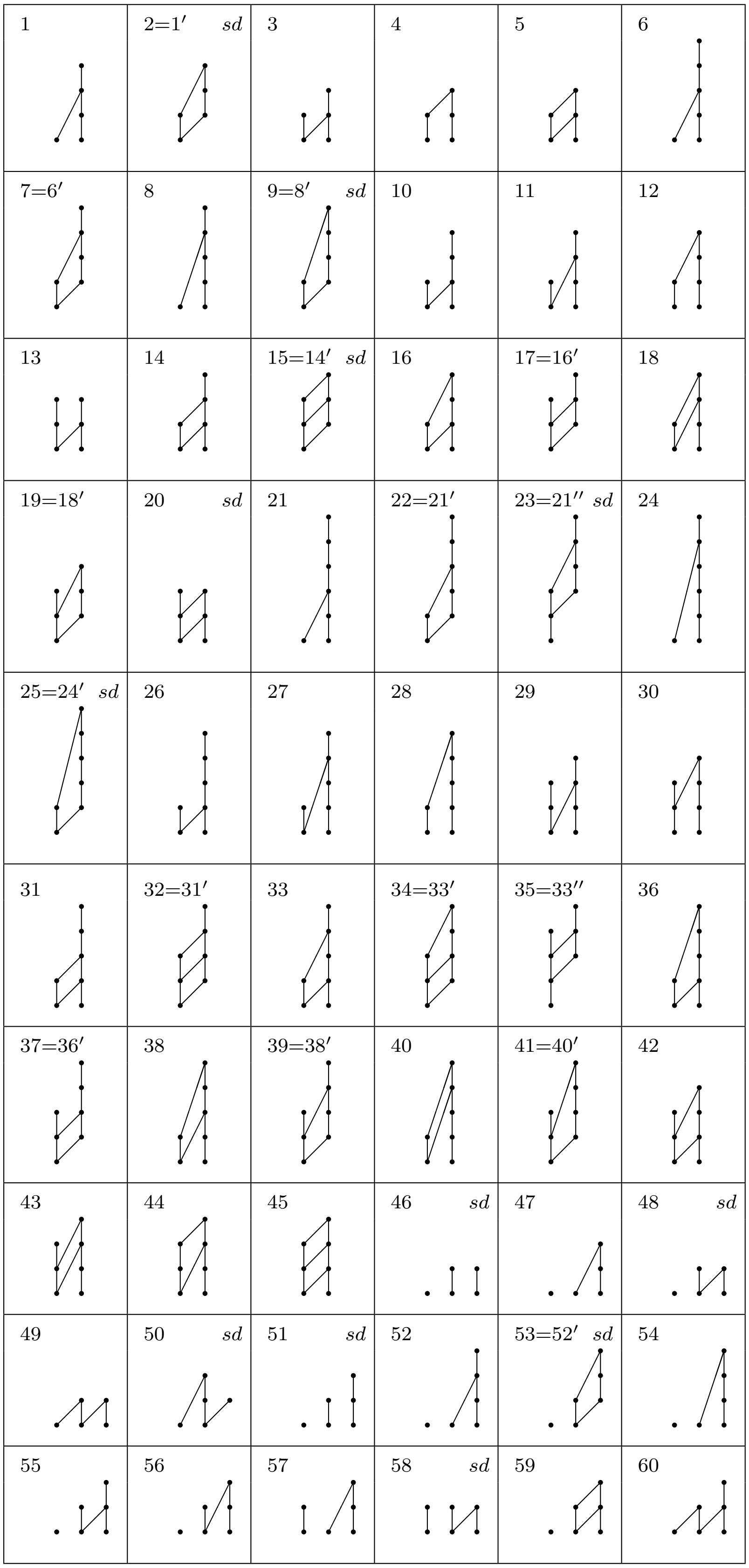
(4.2.3) every set $QC(\mathrm{N})$ contains a quasi-chained poset of width 3.

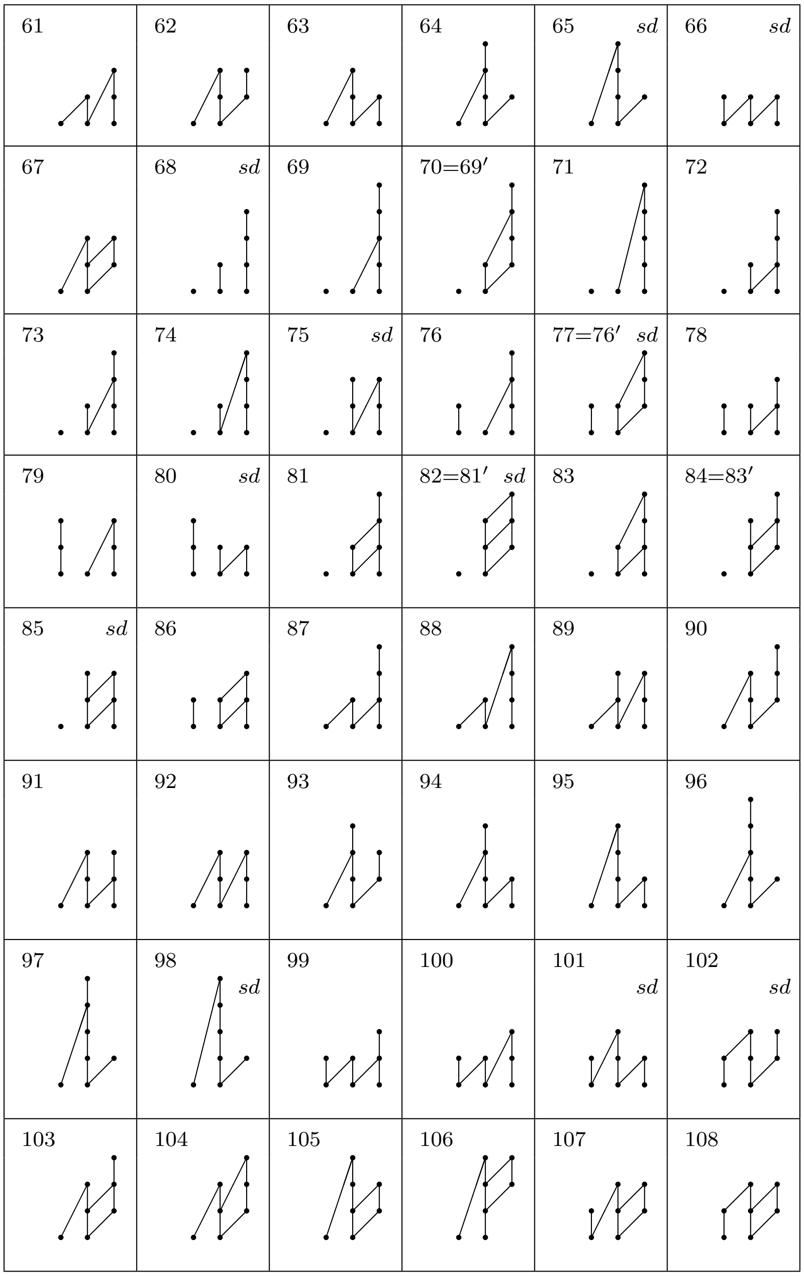
Obviously, the second part of Theorem 3 (for non-serial posets) follows from (4.2.1), Theorem 5 follows from (4.2.2), and the first part of Theorem 6 follows from (4.2.3).

Thus, Theorems 3–6 are proved in both serial and non-serial cases.

The method we proposed allows to establish other properties of positive and not only positive posets.

5. Table of all non-serial positive posets up to isomorphism and duality from the paper [5]. For aesthetic reason, the posets 12 and 13 are arranged in the below table in opposite order.





Some remarks on the table.

In upper right corners the symbol *sd* means that the corresponding poset is self-dual.

If a poset i has width 2 and the table writes $i = j'$, this means that i can be obtained from j by replacing its only maximal point with its only new minimal point. Note that these two posets are (min, max)-isomorphic. The same applies to the case $i = j'' = (j')'$ (it is needed to compare the posets i and j'). If the poset i has width 3 and the table writes $i = j'$, this means that the above applies not to the posets i and j themselves, but to their connected components of width 2. Note that here i and j are (min, max)-isomorphic too. The same applies to the case $i = j''$.

Arbitrary posets S and T, which are obtained from each other using similar operations are called 0-*isomorphic*. And if we remove from the table the posets with numbers $i = j'$ and $i = j''$, we obtain a description of non-serial positive posets up to 0-isomorphism and duality.

6. Conclusions. Recently the first author introduced some concept with consideration of the corresponding examples, which can be considered as the emergence of a new theory on combinatorial properties of posets with respect to minimax equivalence. In this paper we study from such new point of view the posets with positive Tits quadratic form (which are called positive posets). The received results can be generalized to other classes of posets.

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Бондаренко В. М., Стьопочкіна М. В. Про мінімальні мінімаксні системи твірних для додатних частково впорядкованих множин.

Зображення частково впорядкованих множин (скорочено ч. в. множин), введених Л. А. Назаровою і А. В. Ройтером (у матричній формі) в 1972 р., відіграють важливу роль у сучасній теорії зображень та її застосуваннях. Ю. А. Дрозд у 1974 р. довів, що ч. в. множина S має скінченний зображувальний тип над полем тоді і лише тоді, коли її квадратична форма Тітса

$$q_S(z) =: z_0^2 + \sum_{i\in S} z_i^2 + \sum_{i<j,i,j\in S} z_i z_j - z_0 \sum_{i\in S} z_i,$$

є слабко додатною (тобто додатною на множині невід'ємних векторів), але це твердження не є правильним, на відміну від випадку сагайдаків, коли розглядаються додатні форми. Тому ч. в. множини з додатною квадратичною формою Тітса викликали великий інтерес з різних точок зору як аналоги діаграм Динкіна. У 2005 р. автори описали з точністю до ізоморфізму всі множини з додатною квадратичною формою Тітса. Основним методом вирішення цієї проблеми є так званий метод мінімаксної еквівалентності, запропонований першим автором. Нещодавно він представив деяке поняття (а саме, мінімаксної системи твірних) з розглядом відповідних прикладів, які можна розглянути як появу нової теорії, яка досліджує комбінаторні властивості множин відносно мінімаксної еквівалентності.

У цій статті ми вивчаємо з такої нової точки зору ч. в. множини з додатною квадратичною формою Тітса (які називаються додатними).

Ключові слова: додатна квадратична форма, квадратична форма Тітса, додатна ч. в. множина, мінімаксна еквівалентність та ізоморфізм, мінімаксна система твірних.

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Recived 05.11.2024