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LOCAL NEARRINGS WITH ADDITIVE GROUPS OF ORDER 128

The determination of the finite non-abelian p-groups which are the additive groups of local nearrings is an open problem (Feigelstock, 2006). Therefore it is important to determine such groups and to classify some classes of nearrings with identity on these groups, for example, local nearrings. We study local nearrings on 2-generated groups of order 128.

Keywords: local nearring, additive group, 2-generated group.

1. Preliminaries.

Nearrings are generalization of associative rings, in which the additive group can be non-abelian, and addition is connected with multiplication by only one distributive law, left or right. In this sense local nearrings are generalization of local rings.

Clearly every associative ring is a nearring and each group is the additive group of a nearring, but not necessarily of a nearring with identity. However, it is not true that any finite non-abelian group is the additive group of a nearring with identity.

A nearing with an identity is called local if the set of all non-invertible elements forms a subgroup of its additive group. A study of local nearrings was initiated by Maxson [2] who defined a number of their basic properties.

Complicated symbolic computations are being used to solve problems from different areas of mathematics, in particular, to study of algebraic structures. Based on well-known system of computer algebra GAP [3] we construct and investigate one-sided distributive structures (i.e., local nearrings of small orders) with a view of classification of such models.

Boykett and Nöbauer [4] classified all non-abelian groups of order less than 32 that can be the additive groups of a nearring with identity and found the number of non-isomorphic nearrings with identity on such groups (see also GAP package SONATA [5]).

For the researchers in nearrings, the list of all 698 local nearrings of order at most 31 up to isomorphism is provided by the GAP package SONATA; however, classifying nearrings of higher orders is a significant challenge.

2. Local nearrings on 2-generated groups of order 128.

We recall some definitions.

Definition 1. *A non-empty set R with two binary operations "$+$" and "$\cdot$" is a nearring if:*

1) $(R,+)$ is a group with neutral element 0;

2) $(R,\cdot)$ is a semigroup;

3) $x\cdot(y+z)=x\cdot y+x\cdot z$ for all $x,\ y,\ z\in R$.

Such a nearring is called a left nearring. If axiom 3) is replaced by an axiom $(x+y)\cdot z=x\cdot z+y\cdot z$ for all $x,\ y,\ z\in R$, then we get a right nearring.

The group $(R,+)$ of a nearring R is denoted by R^+ and called the *additive group* of R. If in addition $0\cdot x=0$ for all $x\in R$, then the nearring R is called *zero-symmetric*. Furthermore, R is a *nearring with an identity* i if the semigroup $(R,\cdot)$ is a monoid with identity element i.

Definition 2. *A nearring R with identity is called local if the set L of all non-invertible elements of R forms a subgroup of the additive group R^+ and a nearfield, if $L=0$.*

It was found that the additive group of a finite zero-symmetric local nearring is a p-group [2].

There exist 2328 non-isomorphic groups of order $128=2^7$ from which 162 are 2-generated groups: 5 groups are of exponent 64 and only 2 of these groups are the additive groups of local nearrings, 18 groups are of exponent 32 and only 6 of these groups are the additive groups of local nearrings, 65 groups are of exponent 16 and only 16 of these groups are the additive groups of local nearrings, 72 groups are of exponent 8 (unknown the number of the groups which are the additive groups of local nearrings), and 2 groups are of exponent 4 and both of these groups are the additive groups of local nearrings).

Let $[n,i]$ be the i-th group of order n in the SmallGroups library in the computer system algebra GAP. We denote by C_n and Q_n the cyclic and quaternion groups of order n, respectively.

Theorem 1. *The following 2-generated groups of exponent 4 and only they are the additive groups of zero-symetric local nearrings of order 128:*

IdGroup	*Structure Description*	*Number of LNR*
$[128,\ 36]$	$(C_2\times((C_4\times C_2)\rtimes C_2))\rtimes C_4$	>80384
$[128,\ 125]$	$(C_4\times C_4\times C_2)\rtimes C_4$	>35040

Lemma 1. *The following 2-generated groups of exponent 8 are the additive groups of zero-symmetric local nearrings of order 128:*

IdGroup	*Structure Description*	*Number of LNR*
[128, 2]	$((C_8 \times C_2) \rtimes C_4) \rtimes C_2$	> 41184
[128, 4]	$(C_2 \times Q_8) \rtimes C_8$	> 103424
[128, 5]	$(C_8 \times C_2) \rtimes C_8$	> 1536
[128, 6]	$(C_8 \times C_4) \rtimes C_4$	> 73728
[128, 7]	$(C_8 \times C_2) \rtimes C_8$	> 4160
[128, 8]	$(C_4 \rtimes C_8) \rtimes C_4$	> 10240
[128, 12]	$((C_8 \times C_2) \rtimes C_2) \rtimes C_4$	> 1336
[128, 13]	$(C_8 \times C_2) \rtimes C_8$	> 33928
[128, 27]	$(C_8 \rtimes C_4) \rtimes C_4$	> 106240
[128, 38]	$((C_8 \times C_2) \rtimes C_2) \rtimes C_4$	> 80384
[128, 48]	$(((C_8 \times C_2) \rtimes C_2) \rtimes C_2) \rtimes C_2$	> 194080
[128, 49]	$(C_4 \times C_2 \times C_2) \rtimes C_8$	> 191520
[128, 50]	$((C_4 \times C_2) \rtimes C_8) \rtimes C_2$	> 16992
[128, 51]	$(C_2 \times Q_8) \rtimes C_8$	> 16992
[128, 56]	$(C_4 \times C_4) \rtimes C_8$	> 254208
[128, 57]	$(C_4 \times C_4) \rtimes C_8$	> 127488

Question 1. *Are the following 2-generated groups of exponent* 8 *the additive groups of zero-symmetric local nearrings of order* 128*?*

IdGroup	Structure Description
[128, 9]	$(C_8 \times C_2) \rtimes C_8$
[128, 28]	$(C_4 \rtimes C_8) \rtimes C_4$
[128, 126]	$(C_2.((C_4 \times C_2) \rtimes C_2) = (C_2 \times C_2).(C_4 \times C_2)) \rtimes C_4$

Theorem 2. *Only the following groups of order* 128 *and exponent* 16 *are the additive groups of zero-symmetric local nearrings:*

IdGroup	*Structure Description*	*Number of LNR*
[128, 42]	$C_{16} \times C_8$	> 134754
[128, 43]	$C_{16} \rtimes C_8$	> 133866
[128, 44]	$C_8 \rtimes C_{16}$	> 145648
[128, 46]	$((C_{16} \times C_2) \rtimes C_2) \rtimes C_2$	> 24704
[128, 47]	$((C_{16} \times C_2) \rtimes C_2) \rtimes C_2$	252928
[128, 52]	$((C_{16} \rtimes C_2) \rtimes C_2) \rtimes C_2$	> 115840
[128, 53]	$((C_{16} \rtimes C_2) \rtimes C_2) \rtimes C_2$	> 277248
[128, 54]	$(C_4 \times C_2) \rtimes C_{16}$	> 82944
[128, 55]	$(C_4 \times C_2).((C_4 \times C_2) \rtimes C_2) = (C_4 \times C_2).(C_8 \times C_2)$	640
[128, 59]	$C_4.((C_2 \times C_2 \times C_2) \rtimes C_4) = (C_4 \times C_2).(C_8 \times C_2)$	> 13056
[128, 99]	$C_8 \rtimes C_{16}$	> 29248
[128, 102]	$C_8 \rtimes C_{16}$	> 5376
[128, 106]	$(C_{16} \times C_2) \rtimes C_4$	> 2808
[128, 107]	$(C_{16} \times C_2) \rtimes C_4$	> 16460
[128, 108]	$(C_{16} \rtimes C_2) \rtimes C_4$	> 1344
[128, 109]	$(C_{16} \rtimes C_2) \rtimes C_4$	> 2344

Theorem 3. *There exist* 389976 *zero-symetric local nearrings on* 2*-generated additive groups of exponent* 32 *of order* 128*:*

IdGroup	*Structure Description*	*Number of LNR*
[128, 128]	$C_{32} \times C_4$	48968
[128, 129]	$C_{32} \rtimes C_4$	48968
[128, 131]	$(C_{32} \times C_2) \rtimes C_2$	144016
[128, 132]	$(C_{32} \rtimes C_2) \rtimes C_2$	23936
[128, 153]	$C_4 \rtimes C_{32}$	118968
[128, 154]	$C_{16}.D_8 = C_4.(C_{16} \times C_2)$	5120

Theorem 4. *There exist* 1024 *zero-symetric local nearrings on* 2*-generated additive groups of exponent* 64 *of order* 128*:*

IdGroup	*Structure Description*	*Number of LNR*
[128, 159]	$C_{64} \times C_2$	512
[128, 160]	$C_{64} \rtimes C_2$	512

The library of zero-symmetric local nearrings of order 128 on 2-generated groups can be extracted from [7] using GAP and the package LocalNR [6].

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References

1. Feigelstock, S. (2006). Additive Groups of Local Near-Rings. *Comm. Algebra,* 34, 743–747.
2. Maxson, C. J. (1968). On local near-rings. *Math. Z.,* 106, 197–205.
3. The GAP Group, GAP — Groups, Algorithms, and Programming, Version 4.13.0; 2024. `https://www.gap-system.org`
4. Boykett, T. H. H., & Nöbauer C. (1998). *A class of groups which cannot be the additive groups of near-rings with identity.* Contributions to general algebra, 10 (Klagenfurt, 1997). Klagenfurt: Heyn.
5. Aichinger, E., Binder, F., Ecker, Ju., Mayr, P., & Noebauer, C. (2018). SONATA — system of near-rings and their applications. *GAP package, Version 2.9.1.* `https://gap-packages.github.io/sonata/`
6. Raievska, I., Raievska, M., & Sysak, Y. (2024). LocalNR, Package of local nearrings, Version 1.0.4 (GAP package). `https://gap-packages.github.io/LocalNR/`
7. Raievska, I., Raievska, M., & Sysak, Y. (2022). DatabaseEndom128: (v0.2) [Data set]. Zenodo. `https://zenodo.org/records/7225377`

Раєвська І. Ю. Локальні майже-кільця з адитивною групою порядку 128.

Визначення скінченних неабелевих p-груп, які є адитивними групами локальних майже-кілець, є відкритою проблемою (Feigelstock, 2006). Тому важливо визначити такі групи та класифікувати деякі класи майже-кілець з одиницею на цих групах, наприклад, локальні майже-кільця. В статті ми досліджуємо локальні майже-кільця на 2-породжених групах порядку 128.

Ключові слова: локальне майже-кільце, адитивна група, 2-породжена група.

Список використаної літератури

1. Feigelstock S. Additive Groups of Local Near-Rings. *Comm. Algebra.* 2006. Vol. 34. P. 743–747.
2. Maxson C. J. On local near-rings. *Math. Z.* 1968. Vol. 106. P. 197–205.
3. The GAP Group, GAP — Groups, Algorithms, and Programming, Version 4.13.0; 2024. `https://www.gap-system.org`
4. Boykett T. H. H., Nöbauer C. A class of groups which cannot be the additive groups of near-rings with identity. Contributions to general algebra, 10 (Klagenfurt, 1997). Klagenfurt : Heyn, 1998. P. 89–99.
5. Aichinger E., Binder F., Ecker Ju., Mayr P., and Noebauer C. (2018). SONATA — system of near-rings and their applications. *GAP package, Version 2.9.1.* `https://gap-packages.github.io/sonata/`
6. Raievska, I., Raievska, M., and Sysak, Y. LocalNR, Package of local nearrings, Version 1.0.4. 2024. (GAP package). `https://gap-packages.github.io/LocalNR/`
7. Raievska, I., Raievska, M., and Sysak, Y. DatabaseEndom128: (v0.2) [Data set]. Zenodo, 2022. `https://zenodo.org/records/7225377`

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