UDC 512
DOI https://doi.org/10.24144/2616-7700.2025.46(1).13-17

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THE GENERATION PROBLEM FOR INVERTIBLE MEALY AUTOMATA

An invertible Mealy automaton A over an input-output alphabet X generates a group $G(A)$ through its action on words over X. We prove that the following problem is algorithmically undecidable: given two invertible Mealy automata, decide whether they generate the same automaton group. Moreover, we construct an invertible Mealy automaton A for which the following problem is undecidable: given an invertible Mealy automaton B, decide whether B generates $G(A)$ or not. We also show that the automaton subgroup problem and the intersection triviality problem are undecidable.

Keywords: Mealy automaton, automaton group, generation problem, subgroup problem, undecidable problem.

1. Introduction. Decision problems in group theory address various computational aspects of groups. Among the classical group-theoretic decision problems are the word problem, the conjugacy problem, the isomorphism problem, the membership problem, and the generation problem. Although all of these problems are undecidable for finitely presented groups, positive results are known for various classes of groups (see, for example, the survey [5]).

In this paper, we consider groups associated with Mealy automata, known as automaton groups. An invertible Mealy automaton A (henceforth, simply an automaton) over the same input-output alphabet X generates a group $G(\mathsf{A})$ via its action on the set of words over X. Decision problems for automaton groups have been intensively studied. The word problem in every automaton group is decidable. In [7], an automaton group with an undecidable conjugacy problem was constructed. Using the same approach, the undecidability of the isomorphism problem for automaton groups can be shown. In [4], the undecidability of the order problem for automaton groups was proved. Furthermore, the group $F_2 \times F_2$, which can be realized by an automaton, has undecidable generation and membership problems (see [5]). Several problems are known to be decidable for the class of bounded automata (see [1, 2]).

In contrast to classical group-theoretic decision problems, where the input is typically a word over group generators, for automaton groups the input can be given by automata. Specifically, we consider the uniform automaton generation problem: Given two automata A and B, decide whether $G(\mathsf{A}) = G(\mathsf{B})$ or not. For a fixed automaton A, we also consider *the automaton generation problem for A*: Given an automaton B, decide whether B generates $G(\mathsf{A})$ or not. We prove that these problems are undecidable.

Theorem 1. *There exists an invertible Mealy automaton with undecidable automaton generation problem.*

The proof of Theorem 1 is constructive; we exhibit the automaton A that generates the group $\mathbb{Z}^4 \rtimes (F_2 \times F_2)$, using the construction given in [3,7], and show that it satisfies the theorem.

Another problem we consider is *the automaton subgroup problem for A*: Given an automaton B, decide whether $G(\mathsf{B})$ is a subgroup of $G(\mathsf{A})$ or not.

Theorem 2. *There exists an invertible Mealy automaton with undecidable automaton subgroup problem.*

The triviality problem — given an automaton A, decide whether $G(\mathsf{A})$ is trivial or not — is decidable via automata minimization. Next, we address *the automaton intersection triviality problem*: Given two automata A and B, decide whether $G(\mathsf{A}) \cap G(\mathsf{B})$ is trivial or not.

Theorem 3. *The automaton intersection triviality problem is undecidable.*

2. Mealy automata and groups. Let X be a finite alphabet, and X^* be the free monoid generated by X. The elements of X^* are words over X. The length of a word w is the number of letters in it.

An automaton over X is a tuple $\mathsf{A} = (S, X, \pi : S \times X \to X \times S)$, where S is a finite set of states and π is the transition-output function. We can represent A as the directed labeled graph whose vertices are the states, and the edges are determined by the transition-output function as follows:

$$s \xrightarrow{x/y} t \quad \text{whenever} \quad \pi(s, x) = (y, t).$$

An automaton A is called invertible if, for every state $s \in S$ and output $y \in X$, there exist an input $x \in X$ and a state $t \in S$ such that $\pi(s, x) = (y, t)$.

Let A be an automaton and s be its state. The automaton A initialized at s processes an input word $x_1 x_2 \dots x_n \in X^*$ and produces an output word $y_1 y_2 \dots y_n \in X^*$ of the same length. This defines a transformation $\mathsf{T}_s : X^* \to X^*$ such that $\mathsf{T}_s(x_1 x_2 \dots x_n) = y_1 y_2 \dots y_n$ if there exists a directed path in A of the form

$$s = s_1 \xrightarrow{x_1/y_1} s_2 \xrightarrow{x_2/y_2} s_3 \xrightarrow{x_3/y_3} \dots \xrightarrow{x_n/y_n} s_{n+1} = t.$$

Let $\mathsf{A} = (S, X, \pi)$ be an invertible automaton. Then, for every $s \in S$, the transformation $\mathsf{T}_s : X^* \to X^*$ is invertible. The group generated by the transformations T_s for $s \in S$ under composition is called the automaton group $G(\mathsf{A})$ generated by A. By construction, an automaton group is a subgroup of the symmetric group $\mathsf{Sym}(X^*)$.

3. The proofs. Our strategy is to reduce the problems to the following known undecidable problems in the direct product of two free groups of rank two.

Theorem 4 (Miller [5])**.** *1) The generation problem in the group $F_2 \times F_2$ is undecidable.*

2) The group $F_2 \times F_2$ contains a finitely generated subgroup H such that the generalized word problem for H in $F_2 \times F_2$ — given a word w over the generators of $F_2 \times F_2$, decide whether $w \in H$ or not — is undecidable.

3) The intersection triviality problem in the group $F_2 \times F_2$ is undecidable.

The group $F_2 \times F_2$ can be generated by an automaton (see [6, Sect. 1.10.4]), but this does not immediately yield a proof. The difficulty lies in the fact that the states of a given automaton represent a specific set of group elements. Decidability might hold for such specific sets; for instance, every initial automaton representing a nontrivial group element might contain states for all group generators.

Instead, we will use the automaton realization of the affine group $\mathsf{Aff}(\mathbb{Z}^n) = \mathbb{Z}^n \rtimes GL_n(\mathbb{Z})$ constructed in [3] (see also [7]). The elements of $\mathsf{Aff}(\mathbb{Z}^n)$ are pairs (v, M) for $v \in \mathbb{Z}^n$ and $M \in GL_n(\mathbb{Z})$, and the operation is

$$(v, M) \cdot (u, N) = (v + Mu, MN), \text{ where } v, u \in \mathbb{Z}^n \text{ and } M, N \in GL_n(\mathbb{Z}).$$

The group $\mathsf{Aff}(\mathbb{Z}^n)$ acts on $\mathbb{Z}^n$ by affine transformations $(v, M) : x \mapsto v + Mx$, $x \in \mathbb{Z}^n$. For $l \geq 0$, taking modulo 2^l induces an action on $\mathbb{Z}^n/2^l\mathbb{Z}^n$. The elements of $\mathbb{Z}/2^l\mathbb{Z}$ are uniquely represented as binary words of length l, and thus $\mathbb{Z}/2^l\mathbb{Z}$ can be identified with $\{0,1\}^l$. Similarly, elements of $\mathbb{Z}^n/2^l\mathbb{Z}^n$ are represented by words of length l over $X = \{0,1\}^n$, and thus $\mathbb{Z}^n/2^l\mathbb{Z}^n$ can be identified with X^l. In this way, we get an action of $\mathsf{Aff}(\mathbb{Z}^n)$ on the space X^*. (Equivalently, we may consider an affine action on $\mathbb{Z}_2^n$, the set of n-tuples of dyadic integers, and use the dyadic representation of its elements.)

It was proved in [3,7] that the action $(\mathsf{Aff}(\mathbb{Z}^n), X^*)$ can be realized by automata. For $v \in \mathbb{Z}^n$, let $\mathsf{Mod}(v) \in X$ and $\mathsf{Div}(v) \in \mathbb{Z}^n$ denote its remainder and quotient, respectively, when divided by two, here $v = \mathsf{Mod}(v) + 2\mathsf{Div}(v)$. Let $M \in GL_n(\mathbb{Z})$ and $v \in \mathbb{Z}^n$. We construct an automaton $\mathsf{A}_{(v,M)}$ representing the action of (v, M) on X^*. The states are of the form (u, M), where u ranges over a finite subset of $\mathbb{Z}^n$. We begin with the state (v, M) and iteratively construct new states and arrows by the following rule:

$$(v, M) \xrightarrow{x/y} (u, M), \text{ where } y = \mathsf{Mod}(v + Mx) \text{ and } u = \mathsf{Div}(v + Mx).$$

The resulting automaton $\mathsf{A}_{(v,M)}$ has at most $2(a+b)^n$ states, where a is the maximal absolute row sum of M and b is the maximal absolute entry in v. The automaton transformation of X^* induced by a state (u, M) is exactly the transformation induced by the element $(u, M) \in \mathsf{Aff}(\mathbb{Z}^n)$. In particular, the group generated by the automaton $\mathsf{A}_{(v,M)}$ coincides with the group generated by the affine transformation (u, M), where (u, M) ranges over the states. The crucial property of $\mathsf{A}_{(v,M)}$ that we will exploit is that all states share the same linear part M. When v is the zero vector, we write A_M instead of $\mathsf{A}_{(0,M)}$.

We are ready to prove our results.

Proof of Theorem 1. The free group F_2 can be generated by matrices

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

By using two diagonal embeddings into $GL_4(\mathbb{Z})$, we obtain the set of four matrices $S = \{M_1, M_2, M_3, M_4\}$ generating $F_2 \times F_2$. Define the automaton A as the union of the automata A_M for $M \in S$. The automaton A essentially consists of four copies of the automaton described in Figure 1, where each copy corresponds to a specific embedding of $\{0,1\}^2$ into $X = \{0,1\}^4$, with a trivial extension to the remaining

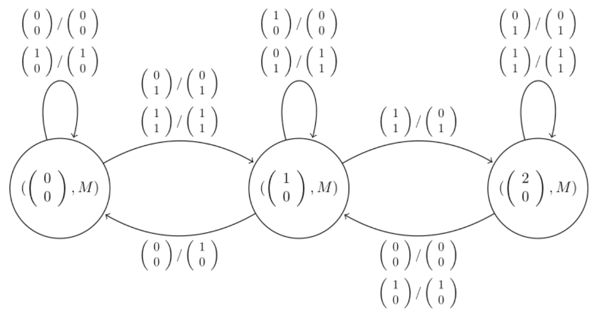

Figure 1. The automaton A_M for $M = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

elements of X. Notice that the group $G(\mathsf{A})$ generated by A contains standard basis vectors of $\mathbb{Z}^4$. Then, by construction, $G(\mathsf{A}) = \mathbb{Z}^4 \rtimes (F_2 \times F_2)$.

We show that the automaton A fulfills the requirement of the theorem. Given words $w_1, \ldots, w_n$ over the generators S of $F_2 \times F_2$, consider the corresponding automata $\mathsf{A}_{w_1}, \ldots, \mathsf{A}_{w_n}$. Note that the linear part of every state of A_{w_i} is w_i. Let the automaton B be the union of these automata together with automata $\mathsf{A}_{(v,E)}$ for the standard basis vectors v of $\mathbb{Z}^4$. Then the automaton B generates the group $G(\mathsf{B}) = \mathbb{Z}^4 \rtimes \langle w_1, \ldots, w_n \rangle$. Therefore, $w_1, \ldots, w_n$ generate $F_2 \times F_2$ if and only if B generates $G(\mathsf{A})$. Since the first problem is undecidable by Theorem 4, the automaton generation problem for A is undecidable.

Proof of Theorem 2. Let H be a finitely generated subgroup of $F_2 \times F_2$ with an undecidable generalized word problem in $F_2 \times F_2$. Let A be an automaton generating the group $G(\mathsf{A}) = \mathbb{Z}^4 \rtimes H$. Given a word w over the generators of $F_2 \times F_2$, construct the automaton A_w as above. Let B_w be the union of A_w and automata $\mathsf{A}_{(v,E)}$ for the standard basis vectors v of $\mathbb{Z}^4$. Note that the linear part of every state of B_w is either w or trivial. Then B_w generates the group $G(\mathsf{B}_w) = \mathbb{Z}^4 \rtimes \langle w \rangle$. Therefore, $G(\mathsf{B}_w)$ is a subgroup of $G(\mathsf{A})$ if and only if $w \in H$. The result follows.

Proof of Theorem 3 follows a similar argument.

4. Conclusions and prospects for further research. We established the undecidability of several fundamental problems for groups generated by invertible Mealy automata. While these problems are undecidable in general, they may be solvable for certain classes of Mealy automata. Notably, the generation problem is decidable for the automaton generating the famous Grigorchuk group and for any automaton generating a contracting regular branch group. An interesting open question remains whether the generation problem is decidable for every bounded automaton.

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Бондаренко Є. В. Проблема породження для оборотних автоматів Мілі.

Оборотний автомат Мілі A над алфавітом входів-виходів X породжує групу $G(A)$ своєю дією на словах над X. Ми доводимо, що наступна проблема алгоритмічно нерозв'язна: за заданими двома оборотними автоматами Мілі, визначити, чи породжують вони одну й ту ж автоматну групу. Крім того, ми будуємо оборотний автомат Мілі A, для якого наступна проблема є нерозв'язною: за заданим оборотним автоматом Мілі B, визначити, чи породжує B групу $G(A)$. Ми також доводимо, що проблема автоматної підгрупи та проблема тривіальності перетину є нерозв'язними.

Ключові слова: автомат Мілі, автоматна група, проблема породження, проблема підгрупи, нерозв'язна проблема.

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Recived 15.02.2025