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A GENERALIZATION OF THE EXPONENTIATED G DISTRIBUTION FAMILY WITH APPLICATIONS TO THE BURR DISTRIBUTION AND THE JOHNSON S_U DISTRIBUTION

A new general approach for building the distribution families is proposed. This approach is quite simple but it allows to create a very extensive range of distributions (a so-called MET-G family). Two particular cases, the monotone exponent transformation Burr XII and the monotone exponent transformation Johnson S_U are studied in detail. Flexibility of the new distributions is demonstrated by fitting them to real data.

Keywords: exponentiated G family, Burr distribution, Johnson family of distributions, maximum likelihood estimation, goodness of fit.

1. Introduction. Creation of new distribution families has attracted considerable attention during recent years. One well-known family is the exponentiated G family (see [1]). This family and its various extensions and generalizations were often used for extension of existing distributions (such new distributions usually exhibit improved flexibility and allow to describe real data more precisely). Models based on the exponentiated G family or the exponentiation of a distribution function include, in particular, the exponentiated generalized G distributions (see [2]), the weighted exponentiated family of distributions by Ahmad et al. (see [3]), the exponentiated generalized alpha power G family (see [4]), the exponentiated generalized Marshall–Olkin family (see [5]), the generalized exponentiated class of distributions by Rezaei et al. (see [6]), the transmuted exponentiated generalized-G family (see [7]), the Marshall-Olkin exponentiated generalized G family (see [8]), to mention but a few.

We propose a new wide-ranging extension of the exponentiated G family, the monotone exponent transformation G (MET-G) family. We study theoretical properties of the MET-G family for two concrete G distributions (the Burr XII distribution and the Johnson S_U distribution) and illustrate the usefulness of the two corresponding MET-G subfamilies by fitting them to real data.

2. The MET-G family. Let G be a distribution which is concentrated on an interval $I \subset \mathbb{R}^1$. Denote its cdf and pdf by $G(x)$ and $g(x)$ correspondingly.

Definition 1. *Suppose that $a(x)$ is a non-increasing differentiable function on I, $a(x) > 0$. The cdf of the MET-G distribution is defined by*

$$F(x) = G(x)^{a(x)}. \tag{1}$$

Remark 1. *If $a(x) = a$ then the MET-G family coincides with the exponentiated G family.*

Remark 2. *It is worth noting that due to very mild restrictions on $a(x)$ the MET-G family is extremely diverse for a fixed G.*

It follows immediately from Definition 1 that the pdf of the MET-G distribution is

$$f(x) = G(x)^{a(x)} \left(a'(x) \ln G(x) + a(x) \frac{g(x)}{G(x)} \right). \tag{2}$$

The quantile function is not available in closed form in general case.

The nth moment of the MET-G distribution can be expressed as

$$\mu'_n = \int_{-\infty}^{\infty} x^n G(x)^{a(x)} \left(a'(x) \ln G(x) + a(x) \frac{g(x)}{G(x)} \right) dx.$$

Let us acquire the densities of order statistics of the MP-G distribution. Using the formula for the pdf of the k-th order statistic (see e.g. [9]), we obtain that the density of the k-th order statistic of MP-G can be expressed as

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} G(x)^{k \cdot a(x)} \left(1 - G(x)^{a(x)}\right)^{n-k} \left(a'(x) \ln G(x) + a(x) \frac{g(x)}{G(x)} \right).$$

3. Monotone exponent transformation Burr XII (METB). Let us investigate the case when the base distribution G is the Burr XII distribution.

The three-parameter Burr XII distribution has the cdf

$$G(x; \alpha, \theta, \gamma) = 1 - \left(1 + \left(\frac{x}{\theta}\right)^{\gamma}\right)^{-\alpha}, \; x > 0 \tag{3}$$

and the density

$$g(x; \alpha, \theta, \gamma) = \alpha \gamma \theta^{-\gamma} x^{\gamma - 1} \left(1 + \left(\frac{x}{\theta}\right)^{\gamma}\right)^{-\alpha - 1}, \; x > 0. \tag{4}$$

We will use the Burr XII distribution in conjunction with $a(x) = \sigma \exp\{-x^{\lambda}\}$ in order to create the monotone exponent transformation Burr XII (METB) distribution. The cdf of the METB distribution is obtained by plugging $G(x)$ from (3) into (1), it equals

$$F(x; \sigma, \lambda, \alpha, \theta, \gamma) = \left(1 - \left(1 + \left(\frac{x}{\theta}\right)^{\gamma}\right)^{-\alpha}\right)^{\sigma \exp\{-x^{\lambda}\}}, \quad x > 0. \tag{5}$$

The pdf of the METB distribution is

$$\begin{aligned} f(x; \sigma, \lambda, \alpha, \theta, \gamma) = \sigma \exp\{-x^{\lambda}\} \left(1 - \left(1 + \left(\frac{x}{\theta}\right)^{\gamma}\right)^{-\alpha}\right)^{\sigma \exp\{-x^{\lambda}\}} \\ \cdot \left(-\lambda x^{\lambda - 1} \ln \left(1 - \left(1 + \left(\frac{x}{\theta}\right)^{\gamma}\right)^{-\alpha}\right) \right. \\ \left. + \alpha \gamma \theta^{-\gamma} x^{\gamma - 1} \left(1 + \left(\frac{x}{\theta}\right)^{\gamma}\right)^{-\alpha - 1} \left(1 - \left(1 + \left(\frac{x}{\theta}\right)^{\gamma}\right)^{-\alpha}\right)^{-1} \right), \; x > 0. \end{aligned} \tag{6}$$

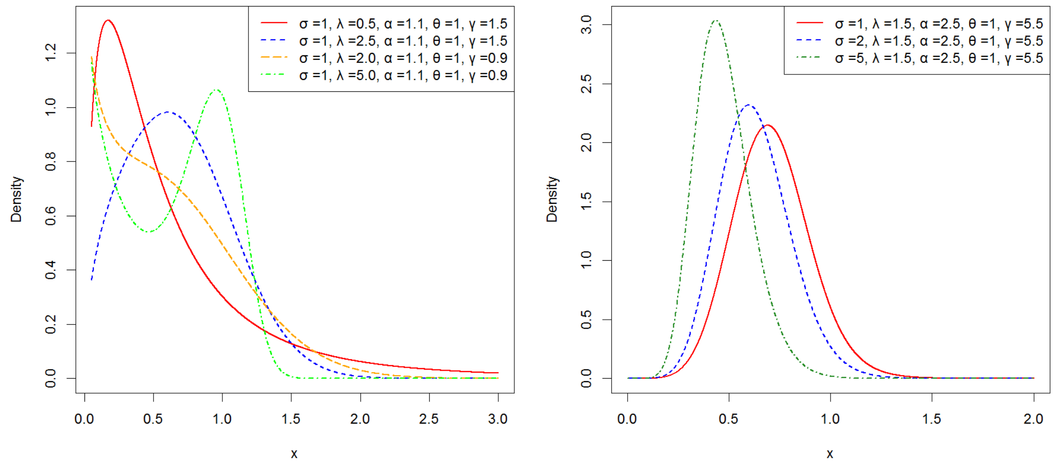

Figure 1. Density plots of the METB distribution.

We will use notations $F(x)$ and $f(x)$ for the left-hand sides of (5) and (6) correspondingly when this is unambiguous.

Fig. 1 displays plots of the pdf of the METB distribution for selected parameter values.

There exists no closed form for the quantile function $Q(u;\sigma,\lambda,\alpha,\theta,\gamma)$ of the METB distribution. However, it can be obtained approximately by numerical inversion of the cdf using a package like Mathematica, MATLAB or R.

The quantile function allows to determine such shape measures as the Bowley's skewness and the Moors' kurtosis. Namely, the Bowley's skewness

$$S_B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)},$$

the Moors' kurtosis

$$K_M = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}.$$

The plots of the Bowley's skewness and Moors' kurtosis of the METB distribution are displayed in Fig. 2 (on the left-hand side and on the right-hand side, respectively).

It is evident that the influence of the values of λ and σ on the skewness and the kurtosis is significant.

The nth moment of a random variable X which has the METB distribution can be obtained as

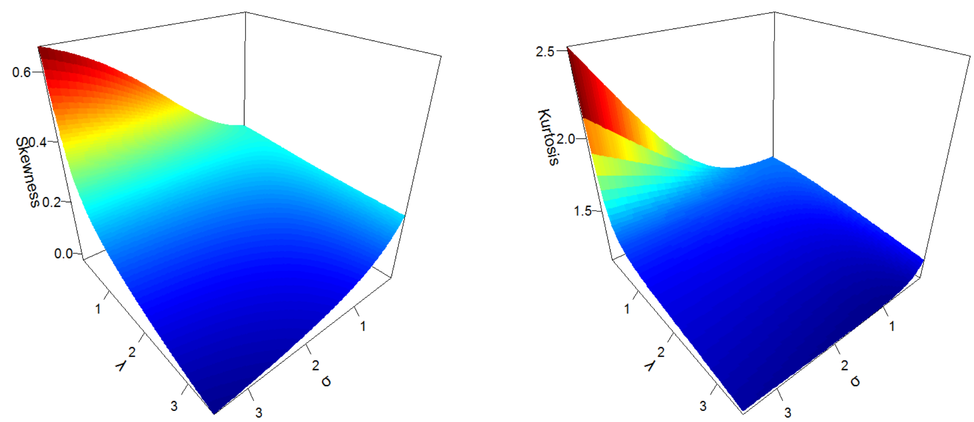

Figure 2. Plots of Bowley's skewness and Moors' kurtosis, METB distribution.

$$\mu'_n = EX^n = \sigma \int_0^\infty x^n \exp\{-x^\lambda\} \left(1 - \left(1 + \left(\frac{x}{\theta}\right)^\gamma\right)^{-\alpha}\right)^{\sigma \exp\{-x^\lambda\}}$$
$$\cdot \left(-\lambda x^{\lambda-1} \ln\left(1 - \left(1 + \left(\frac{x}{\theta}\right)^\gamma\right)^{-\alpha}\right)\right.$$
$$\left. + \alpha\gamma\theta^{-\gamma} x^{\gamma-1} \left(1 + \left(\frac{x}{\theta}\right)^\gamma\right)^{-\alpha-1} \left(1 - \left(1 + \left(\frac{x}{\theta}\right)^\gamma\right)^{-\alpha}\right)^{-1}\right) dx.$$

The nth moment can not be expressed in closed form but it is possible to calculate it using numerical methods.

Let us illustrate the flexibility of the METB distribution using real data. We will consider the data set which consists of the values which resulted from the algorithm SC16 for estimating unit capacity factors, see [10].

The fit of the METB model will be compared to those of the following alternative distributions: the Burr XII distribution, the beta Burr XII distribution (see [11]), the exponentiated Burr XII distribution, and the Gamma-Uniform Burr XII distribution (see [12]).

The values of the MLEs of the METB parameters are given in Table 1. The R software was used for obtaining the estimates numerically.

Table 1.

Maximum likelihood estimates of the METB model

Parameter	$\boldsymbol{\sigma}$	$\boldsymbol{\lambda}$	$\boldsymbol{\alpha}$	$\boldsymbol{\theta}$	$\boldsymbol{\gamma}$
Estimate	29.5884	17.9115	0.2024	0.0151	1.8425

Table 2 provides the log-likelihood and several information criteria.

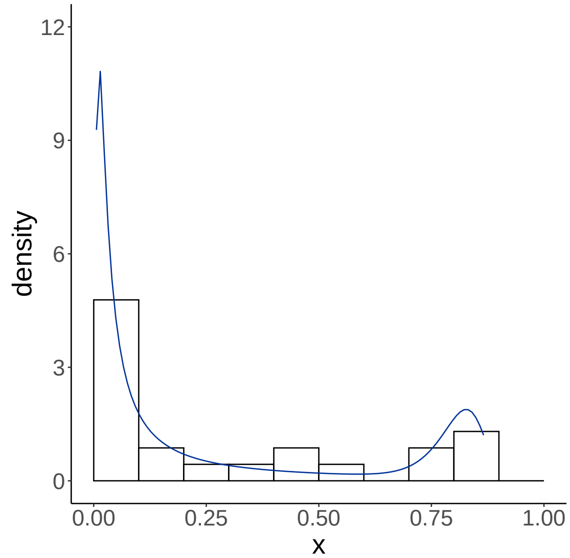

Figure 3. Plot of the estimated pdf
for the values obtained by the algorithm SC16 dataset.

Table 2.

Goodness of fit criteria of the considered models,
values obtained by the algorithm SC16 dataset

Distribution	**Log-likelihood**	**AIC**	**BIC**	**HQIC**
METB	14.945	-19.889	-14.212	-18.461
Burr XII	7.265	-8.530	-5.123	-7.673
Beta-Burr XII	7.753	-5.506	0.172	-4.078
Exponentiated Burr XII	9.296	-10.591	-6.049	-9.449
Gamma uniform Burr XII	9.731	-11.461	-6.919	-10.319

Obviously the results indicate that the METB model corresponds to the better fit since it has the lowest values of the AIC, BIC and HQIC statistics (and the log-likelihood is the highest).

The histogram and the fitted pdf of the METB distribution are depicted in Fig. 3.

4. Monotone exponent transformation Johnson S_U (METJSu). The second special model is the monotone exponent transformation Johnson S_U distribution (METJSu). The base family for this case is the Johnson S_U system, its cdf and pdf are

$$G(x;\gamma,\delta,\xi,\lambda) = \Phi\left(\gamma + \delta \operatorname{arsinh}\left(\frac{x-\xi}{\lambda}\right)\right) \quad (7)$$

and

$$g(x;\gamma,\delta,\xi,\lambda) = \frac{\delta}{\sqrt{2\pi}\sqrt{(x-\xi)^2+\lambda^2}} \exp\left\{-\frac{1}{2}\left(\gamma + \delta \operatorname{arsinh}\left(\frac{x-\xi}{\lambda}\right)\right)^2\right\} \quad (8)$$

correspondingly, where $\delta, \lambda \in (0;\infty)$ and $\Phi(x)$ is the cdf of $N(0;1)$ distribution. We will use the function $a(x) = \exp\{-bx\}$, where $b \in (0;\infty)$. Then the cdf and the pdf of the METJSu model are given by

$$F(x; b, \gamma, \delta, \xi, \lambda) = \left(\Phi \left(\gamma + \delta \operatorname{arsinh}\left(\frac{x-\xi}{\lambda} \right) \right) \right)^{\exp\{-bx\}} \tag{9}$$

and

$$\begin{aligned} f(x; b, \gamma, \delta, \xi, \lambda) = e^{-bx} &\left(\Phi \left(\gamma + \delta \operatorname{arsinh}\left(\frac{x-\xi}{\lambda} \right) \right) \right)^{\exp\{-bx\}} \\ &\cdot \left(-b \ln \Phi \left(\gamma + \delta \operatorname{arsinh}\left(\frac{x-\xi}{\lambda} \right) \right) \right. \\ &\left. + \frac{\delta}{\sqrt{2\pi}} \frac{\exp\left\{ -\frac{1}{2} \left(\gamma + \delta \operatorname{arsinh}\left(\frac{x-\xi}{\lambda} \right) \right)^2 \right\}}{\sqrt{(x-\xi)^2 + \lambda^2}\, \Phi\left(\gamma + \delta \operatorname{arsinh}\left(\frac{x-\xi}{\lambda} \right) \right)} \right). \end{aligned} \tag{10}$$

Plots of the density function of the METJSu distribution for selected parameter values are displayed in Fig.4.

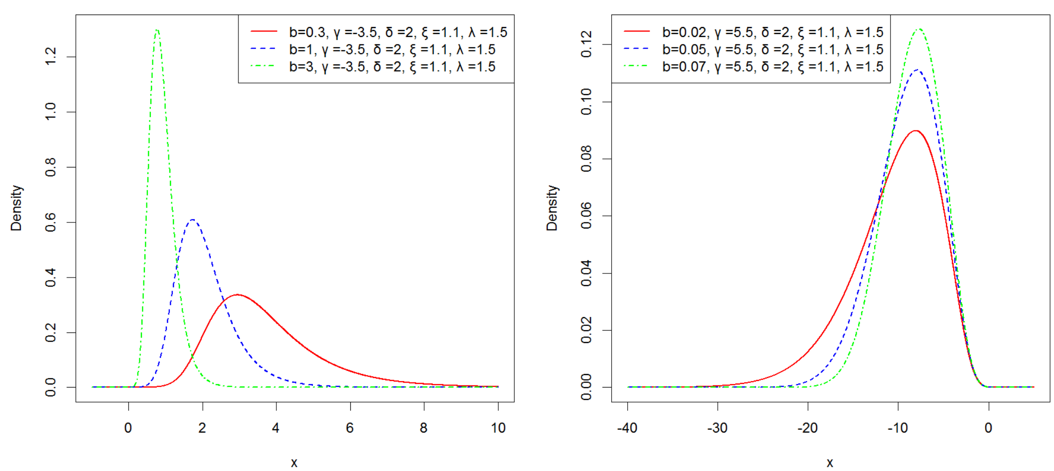

Figure 4. Density plots of the METJSu distribution.

The quantile function of the METJSu distribution can be obtained only numerically. The same is true for the quantile measures, i.e. the Bowley's skewness and the Moors' kurtosis.

We plot the measures S_B and K_M for selected parameter values of the METJSu distribution in Fig. 5 (the upper and the lower figure, respectively).

We fitted the METJSu model to the data set of COP (ConocoPhillips) stock returns from January 13, 1986 till April 4, 1986 (see [13]). The results were compared with the Johnson S_U distribution (JSu), the beta Johnson S_U distribution (BJSu), the exponentiated generalized Johnson S_U distribution (EGJSu), and the generalized hyperbolic distribution (GH).

The histogram and the fitted pdf of the METJSu distribution are shown in Fig. 6.

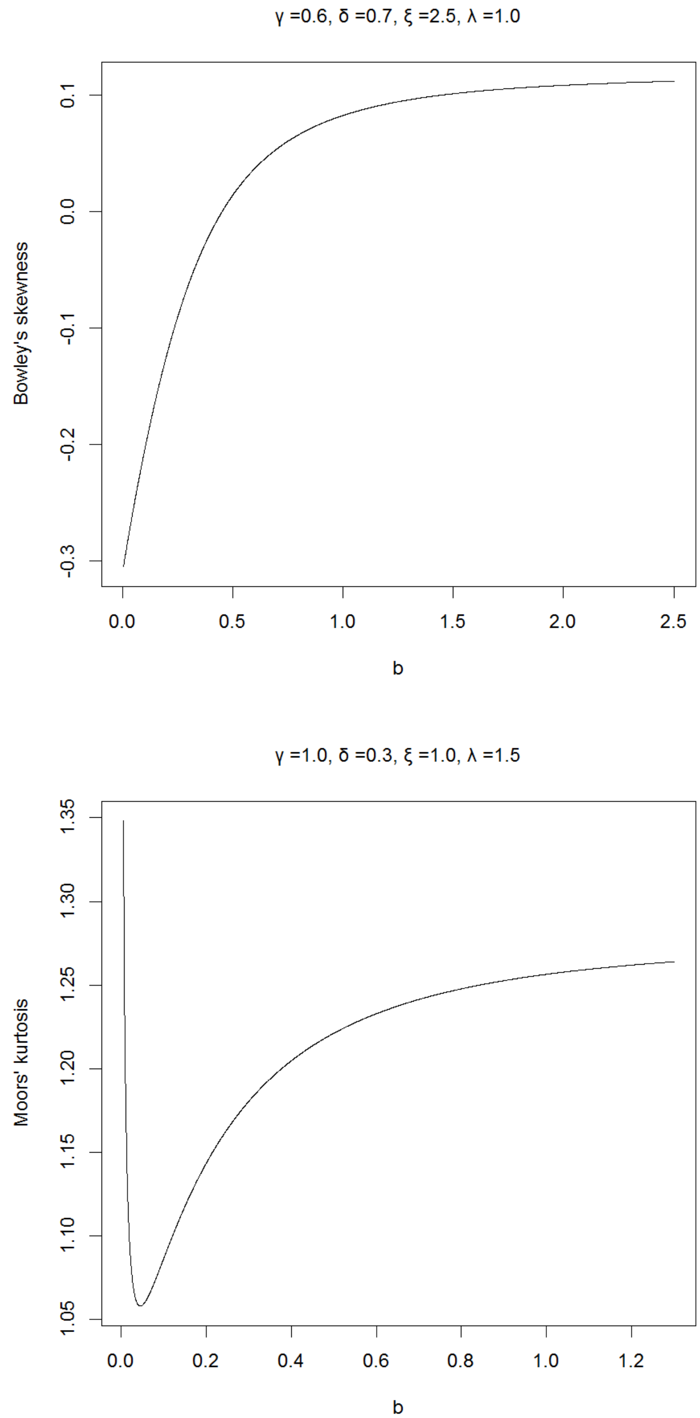

Figure 5. Plots of Bowley's skewness and Moors' kurtosis, METJSu distribution.

Table 3.

Maximum likelihood estimates of the METJSu model

Parameter	b	γ	δ	ξ	λ
Estimate	6.2363	0.4429	0.1207	0.0479	0.0030

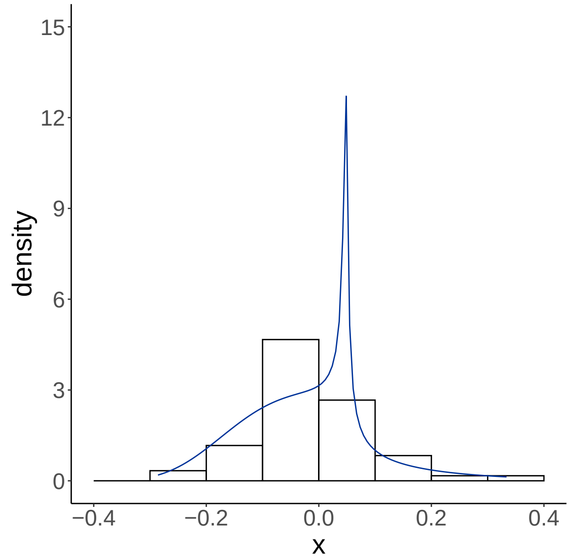

Figure 6. Plot of the estimated pdf for the stock returns dataset

The values of the MLEs of the METJSu parameters are provided in Table 3.

Several goodness of fit statistics are shown in Table 4. The values clearly show that the METJSu distribution is preferable to the competing models.

Table 4.

Goodness of fit criteria of the considered models, the stock returns dataset

Distribution	**Log-likelihood**	**AIC**	**BIC**	**HQIC**
METJSu	56.139	-102.278	-91.806	-98.182
JSu	45.373	-82.746	-74.369	-79.469
BJSu	44.987	-77.973	-65.407	-73.058
EGJSu	45.014	-78.027	-65.461	-73.112
GH	45.471	-80.942	-70.470	-76.845

5. Conclusions. A new distribution family (monotone exponent transformation G family or MET-G), which extends the exponentiated G family, is proposed. The new family encompasses a very wide range of distributions. A detailed analysis of two representatives of the MET-G model, the monotone exponent transformation Burr XII (METB) distribution and the monotone exponent transformation Johnson S_U (METJSu) distribution, is given. Various theoretical properties are studied, in particular, the moments, the Bowley's skewness and the Moors' kurtosis. Two applications to real data prove usefulness and potentiality of the proposed family.

Further studies in this area can include, for instance, generalizations of the monotone exponent transformation G family by including additional parameters and extra transforms.

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Турчин Є. В. Узагальнення “G-піднесеної до ступеня” сім’ї розподілів із застосуваннями до розподілу Берра та розподілу Джонсона S_U.

Запропоновано новий підхід до побудови сімей імовірнісних розподілів. Хоча цей підхід є досить простим, він дозволяє утворити дуже широкий клас розподілів (так звану сім’ю MET-G розподілів). Детально вивчаються два окремі випадки з цієї сім’ї — монотонно показниково перетворений розподіл Берра XII та монотонно показниково перетворений розподіл Джонсона S_U. Гнучкість нових розподілів підтверджено їх підгонкою до реальних наборів даних.

Ключові слова: “піднесена до ступеня” сім’я G розподілів, розподіл Берра, сім’я розподілів Джонсона, оцінки максимальної правдоподібності, адекватність моделі.

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