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SEMIMODULE OF SEMIRING DERIVATIONS

We study derivations of `semirings`, `differential semirings` and the set of all derivations on a semiring. The notion of a `semiring derivation` is traditionally defined as an additive map satisfying the Leibnitz rule. In the paper, we give new examples of semiring derivations, prove some of their properties. We also prove that the set of all derivations on a semiring forms a `semimodule` over its center. We show that the commutator of any two derivations is contained in a `subsemimodule` $V(M)$ of elements of M having additive inverse.

Keywords: semiring derivation, semimodule, semiring, subsemimodule, differential semiring.

1. Introduction and preliminaries. Throughout the paper $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$ the set of non-negative integers.

The notion of a semiring derivation is defined in [3] as an additive map satisfying the Leibnitz rule. In [2] the authors investigated some simple properties of semiring derivations. The objective of this paper is to give new examples of semiring derivations, further explore their properties and investigate the semimodule of the semiring derivations.

Let R be a nonempty set and let $+$ and $\cdot$ be binary operations on R named addition and multiplication respectively. $(R, +, \cdot)$ is called a *semiring* if the following conditions hold:

1) $a + (b + c) = (a + b) + c$ for all $a, b, c \in R$;
2) $(ab)\,c = a\,(bc)$ for all $a, b, c \in R$;
3) $(a + b)\,c = ac + bc$ and $a\,(b + c) = ab + ac$ for all $a, b, c \in R$.

A semiring which is not a ring is called a *proper semiring*. A subset of R closed under addition and multiplication is called a *subsemiring* of R.

The *centre* of a semiring R is a set $Z(R) = \{r \in R | rs = sr \; \forall s \in R\}$. It is a subsemiring of R. An element $r \in Z(R)$ is called *central*.

If not stated otherwise R denotes a semiring. A semiring $(R, +, \cdot)$ is called *additively commutative* if $+$ is a commutative binary operation on R. A semiring $(R, +, \cdot)$ is said to be *multiplicatively commutative* if $\cdot$ is commutative on R. It is said to be *commutative* if both $+$ and $\cdot$ are commutative.

An element $0 \in R$ is called *zero* if $a + 0 = 0 + a = a$ for all $a \in R$. An element $1 \in R$ is called *identity* if $a \cdot 1 = 1 \cdot a = a$ for all $a \in R$. Suppose $1 \neq 0$, otherwise

$R = \{0\}$. Zero $0 \in R$ is called (*multiplicatively*) *absorbing* if $a \cdot 0 = 0 \cdot a = 0$ for all $a \in R$. Note that a multiplicatively absorbing zero $0 \in R$ cannot be additively absorbing unless R contains just one element.

An element $a \in R$ is called a unit if there exists $b \in R$ such that $ab = ba = 1$. The set of all units of R is denoted by $U(R)$. $1 \in U(R)$ and $0 \notin U(R)$.

Many authors define a semiring as additively commutative semirings in the above sense. Moreover, Golan considers additively commutative semirings with zero.

An element $a \in R$ is called *left additively cancellable* if $a + b = a + c$ follows $b = c$ for all $b, c \in R$. An element $a \in R$ is called *right additively cancellable* if $b + a = c + a$ follows $b = c$ for all $b, c \in R$. An element $a \in R$ is called *additively cancellable* if it is both left and right additively cancellable. Denote by $K^+(R)$ the set of all additively cancellable elements of R. A semiring R is called *(left, resp. right) additively cancellative* if every element of R is (left, resp. right) additively cancellable.

Similarly we can define multiplicatively (left, right) cancellable elements and multiplicatively (left, right) cancellative semirings. An element $a \in R$ is called *left multiplicatively cancellable* if $a \cdot b = a \cdot c$ follows $b = c$ for all $b, c \in R$. An element $a \in R$ is called *right multiplicatively cancellable* if $b \cdot a = c \cdot a$ follows $b = c$ for all $b, c \in R$. An element $a \in R$ is called *multiplicatively cancellable* if it is both left and right multiplicatively cancellable. Denote by $K^\times(R)$ the set of all multiplicatively cancellable elements of R. Clearly any unit of R is multiplicatively cancellable and $1 \in U(R) \subset K^\times(R)$. Moreover, $0 \notin K^\times(R)$ and no multiplicatively cancellable element of R is a zero divisor. A semiring R is called *(left, resp. right) multiplicatively cancellative* if every element of R is (left, resp. right) multiplicatively cancellable. $K^\times(R)$ is a submonoid of $(R, \cdot)$.

An element $a \in R$ is called *additively idempotent* if $a + a = a$. Denote by $I^+(R)$ the set of all additively idempotent elements of R. Remind that $K^+(R) \bigcap I^+(R) = \{0\}$. An element $a \in R$ is called *multiplicatively idempotent* if $a \cdot a = a$. A semiring R is called *additively (multiplicatively) idempotent* if every element of R is additively (multiplicatively) idempotent.

Let R be an additively commutative semiring with absorbing zero 0_R. A left semimodule over a semiring R is a commutative additive monoid (M,$+$) with additive identity 0_M together with a scalar multiplication $R \times M \to M$ (sending (r, m) to rm) such that $(rs)m = r(sm)$, $(r + s)m = rm + sm$, $r(m + n) = rm + rn$ and $0_R \cdot m = r \cdot 0_M = 0_M$ for all $r, s \in R$ and $m, n \in M$.

Additively idempotent semirings are of great interest due to their applications. They are widely known as idempotent semirings. For more information on semirings. semimodules, including those with derivations we refer the reader to [3–8].

2. Differential semirings and semiring derivations. Let R be a semiring. A map $\delta \colon R \to R$ is called a *derivation on* R if for any $a, b \in R$ the following conditions hold

1) $\delta(a + b) = \delta(a) + \delta(b)$;
2) $\delta(ab) = \delta(a) b + a\delta(b)$.

The definition of a derivation on an additively commutative semiring with absorbing zero was given by Golan in [3], and generalized for semirings without these conditions by M. Chandramouleeswaran and V. Thiruveni [2]. A semiring R equipped with a derivation δ is called *differential* with respect to the derivation δ, or δ-*semiring*, and denoted by (R, δ) [2]. Denote the set of all derivations on R by $\operatorname{Der} R$.

Example 1. Let R be a semiring. Consider the following subsemiring $S \subseteq M_3(R)$ with respect to ordinary matrix addition and multiplication

$$S = \left\{ \left(\begin{array}{ccc} a & 0 & b \\ 0 & a & b \\ 0 & 0 & a \end{array} \right) \middle| \, a, b \in R \right\}.$$

A map $\delta \colon S \to S$, defined by

$$\delta \left(\left(\begin{array}{ccc} a & 0 & b \\ 0 & a & b \\ 0 & 0 & a \end{array} \right) \right) = \left(\begin{array}{ccc} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right),$$

is a derivation on S. Thus (S, δ) is a differential semiring.

Proposition 1. *If (R, δ) is a differential semiring then the matrix semiring $M_n(R)$ is a differential semiring.*

Proof Define a map $\Delta \colon M_n(R) \to M_n(R)$ elementwise. For a matrix $A = (a_{ij}) \in M_n(R)$ define $\Delta(A) = (\delta(a_{ij}))_{i,j=\overline{1,n}}$. Prove that Δ is a derivation on $M_n(R)$.

Let $A = (a_{ij})_{i,j=\overline{1,n}}$ and $B = (b_{ij})_{i,j=\overline{1,n}}$. Then $\Delta(A + B) = (\delta(a_{ij} + b_{ij}))_{i,j=\overline{1,n}} = (\delta(a_{ij}) + \delta(b_{ij}))_{i,j=\overline{1,n}} = (\delta(a_{ij}))_{i,j=\overline{1,n}} + (\delta(b_{ij}))_{i,j=\overline{1,n}} = \delta((a_{ij}))_{i,j=\overline{1,n}} +$
$+\delta((b_{ij}))_{i,j=\overline{1,n}} = \Delta(A) + \Delta(B)$. Similarly $\Delta(A \cdot B) = \Delta\left(\sum_{k=1}^{n} a_{ik} b_{kj}\right)_{i,j=\overline{1,n}} = \left(\sum_{k=1}^{n} \delta(a_{ik} b_{kj})\right)_{i,j=\overline{1,n}} = \left(\sum_{k=1}^{n} (\delta(a_{ik}) b_{kj} + a_{ik}\delta(b_{kj}))\right)_{i,j=\overline{1,n}} =$
$= \left(\sum_{k=1}^{n} \delta(a_{ik}) b_{kj}\right)_{i,j=\overline{1,n}} + \left(\sum_{k=1}^{n} a_{ik}\delta(b_{kj})\right)_{i,j=\overline{1,n}} = \Delta(A) \cdot B + A \cdot \Delta(B)$.

Example 2. Consider the following example [1] of a two-element semiring $S_2 = \{0, 1\}$ under addition and multiplication defined by the following tables

$+$	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

Semiring S_2 is additively commutative, additively idempotent but not additively cancellative, and zero is not absorbing. It is easily seen that the identity map $\delta_1 = id_R$ ($id_R(a) = a$ for all $a \in R$) and the map $\delta_2 \colon S_2 \to S_2$, given by $\delta_2(a) = 1$ for all $a \in S_2$, are derivations on S_2. Here $\operatorname{Der} S_2 = \{\delta_1, \delta_2\}$. Hence it is not generally true that $\delta(0) = \delta(1) = 0$ in a semiring. Note that the zero map is not a derivation on S_2. This motivates the following easy-to-check proposition

Proposition 2. *If R contains absorbing (left and right) zero then the zero map $0 \colon R \to R$ (i. e. $0(r) = 0$ for all $r \in R$) is a derivation on R.*

If a zero map $0 \colon R \to R$ is a derivation on R it is called a *trivial derivation.* Every semiring with absorbing zero is differential with respect to trivial derivation. Obviously, every ring is differential as a semiring with respect to trivial derivation.

Proposition 3. *The identity map $id_R \colon R \to R$ is a derivation on R in each of the following cases:*

1) R is additively cancellative and $R^2 = (0)$;
2) R is an additively idempotent semiring.

Proof. The identity map is obviously additive. Let $a, b \in R$. The equality $id_R(ab) = id_R(a) \cdot b + a \cdot id_R(b)$ holds if and only if $ab = ab + ab$. The second part is shown by Golan [3] for semirings in his sense.

Clearly, for an additively idempotent semiring R with $1 \neq 0$ there exists a derivation δ on R such that $\delta(1) = 1$. This does not agree (contradicts) to a well-known fact for differential rings with $1 \neq 0$ that 1 is constant with respect to any ring derivation.

Proposition 4. *Let R be a semiring with absorbing zero. For any semiring derivation $\delta\colon R \to R$ we have $\delta(0) = 0$.*

Proof. It follows from $\delta(0) = \delta(0 \cdot 0) = \delta(0) \cdot 0 + 0 \cdot \delta(0) = 0 + 0 = 0$.

Proposition 5. *If R is additively cancellative semiring, $\delta\colon R \to R$ is a derivation then $\delta(1) = 0$ and $\delta(0) = 0$.*

Proof. Since R is additively cancellative, it follows from $\delta(1) = \delta(1 \cdot 1) = \delta(1) \cdot 1 + 1 \cdot \delta(1) = \delta(1) + \delta(1)$ that $\delta(1) = 0$. Similarly, from $\delta(0) = \delta(0 + 0) = \delta(0) + \delta(0)$ we have $\delta(0) = 0$.

Proposition 6. *If $\delta : R \to R$ is a semiring derivation R such that $\forall a \in R\ \ 2a = 0$, then $\delta(1) = 0$ and $\delta(0) = 0$.*

Proof. By the product rule, $\delta(1) = \delta(1 \cdot 1) = \delta(1) \cdot 1 + 1 \cdot \delta(1) = \delta(1) + \delta(1) = 2\delta(1)$. By the sum rule $\delta(0) = \delta(0 + 0) = \delta(0) + \delta(0) = 2\delta(0)$. If R is a semiring such that $\forall a \in R\ \ 2a = 0$, then $\delta(1) = 0\ \delta(0) = 0$.

Definition. Call a semiring (with absorbing zero) *differentially trivial* if $Der(R) = \{0\}$.

Example 3. Find all derivations on the semiring $\mathbb{N}_0$. Let δ be any derivation on R. Since a proper semiring $\mathbb{N}_0$ is additively cancellative (see [3], p. 49) for all $n \in \mathbb{N}$ we have $\delta(n) = \delta\left(\underbrace{1 + 1 + \ldots + 1}_{n}\right) = n \cdot \delta(1) = 0$. Hence, $\mathbb{N}_0$ is a differentially trivial semiring.

Example 4. The semiring $(\mathbb{N}_0 \bigcup \{\infty\}, +, \cdot)$ obtained from $(\mathbb{N}_0, +, \cdot)$ by adjoining a doubly absorbing element ∞ is not differentially trivial. The map

$$\delta(a) = \begin{cases} \infty, & a = \infty; \\ 0, & a \in \mathbb{N}_0. \end{cases}$$

is a nontrivial derivation on $\mathbb{N}_0 \bigcup \{\infty\}$. Note that it is not additively cancellative.

Definition. An element $r \in R$ is called *constant* under the derivation δ if $\delta(r) = 0$.

Clearly, 0 and 1 are constants in any additively cancellative semiring containing these elements. Any natural number is a constant under the trivial derivation.

Example 5. In a semiring $S = \left\{ \left(\begin{array}{cc} a & b \\ 0 & c \end{array} \right) \middle| \, a, b, c \in R \right\}$ with respect to ordinary addition and multiplication under the derivation $d\colon S \to S$ given by $d\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ for each $a, b, c \in R$, $a \neq 0$ $d\left(\left(\begin{array}{cc} a & 0 \\ 0 & c \end{array} \right)\right) = \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)$. Hence, each diagonal matrix of S is constant.

Example 6. Matrices $\begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are constant under the derivation in the Example 1.

Proposition 7. *Let R be additively cancellative semiring. Then every additively idempotent element $a \in R$ is a constant with respect to any derivation δ on R.*

Proof. Since R is additively cancellative $\delta(a) = \delta(a + a) = \delta(a) + \delta(a)$ follows $\delta(a) = 0$.

A derivation $\delta\colon R \to R$ is said to be *commuting* $r \cdot \delta(r) = \delta(r) \cdot r$ for all $r \in R$. .

Proposition 8. *Let R be additively cancellative entire semiring. Then every multiplicatively idempotent element $a \in R$ is a constant with respect to any commuting derivation δ.*

Proof. Let $a \in R$ be any multiplicatively idempotent element. Suppose $a \neq 0$; otherwise clear. Taking the derivative $\delta(a) = \delta(a \cdot a) = \delta(a) \cdot a + a \cdot \delta(a)$. Multiplying by a on the left $a \cdot \delta(a) = a \cdot \delta(a) \cdot a + a^2 \cdot \delta(a)$. By idempotency and additive cancelativeness we get $a \cdot \delta(a) = a \cdot \delta(a) \cdot a + a \cdot \delta(a)$ and $a \cdot \delta(a) \cdot a = 0$. By commutativity and idempotency we have $a \cdot \delta(a) = 0$. Since R contains no zero divisors and $a \neq 0$, then $\delta(a) = 0$.

Note that in the additively cancellative semiring the only additively idempotent element is 0. It is clearly constant.

A subsemiring S of a differential semiring (R, δ) is called *differential* if $\delta(S) \subseteq S$.

Proposition 9. *Let (R, δ) be an additively cancellative additively commutative differential semiring with the center $Z(R)$. Then $Z(R)$ is a differential subsemiring of R.*

Proof. Let $a \in Z(R)$, $b \in R$. We have $\delta(ab) = \delta(ba)$. Then $\delta(ab) = \delta(a) \cdot b + a \cdot \delta(b) = \delta(a) \cdot b + \delta(b) \cdot a$. On the other hand $\delta(ba) = \delta(b) \cdot a + b \cdot \delta(a)$. Since R is additively cancellative and additively commutative we have $\delta(a) \cdot b = b \cdot \delta(a)$. Hence, $\delta(a) \in Z(R)$.

Proposition 10. *The set $C^\delta(R) = \{a \in R \,|\, \delta(a) = 0\}$ of all constants of R with respect to the derivation δ forms a differential subsemiring of R.*

Proof. It is shown in [2] that $C^\delta(R)$ is a subsemiring of R. If $a \in C^\delta(R)$ then $\delta(a) = 0 \in C^\delta(R)$. Hence, $C^\delta(R)$ is closed under δ.

Corollary 1. *The set $I^+(R)$ of additively idempotent elements of the additively cancellative semiring R is a differential semiring.*

Example 7. For a semiring S_2 from Example 2 $C^{\delta_1}(S_2) = (0)$, but $C^{\delta_2}(S_2) = \varnothing$. Also $C^\delta(\mathbb{N}_0 \bigcup \{\infty\}) = \mathbb{N}_0$.

Proposition 11. *If F is a δ-semifield then the semiring of constants C^δ is a δ-semifield.*

Proof. Let $a \in C^\delta(R)$, $a \neq 0$. Then $\delta(a) = 0$ and $a \in F$ follows $a^{-1} \in F$ and $a \cdot a^{-1} = 1$. Then $0 = \delta(1) = \delta(a \cdot a^{-1}) = \delta(a) \cdot a^{-1} + a \cdot \delta(a^{-1}) = a \cdot \delta(a^{-1})$. Thus $a \cdot \delta(a^{-1}) = 0$. Hence $\delta(a^{-1}) = 0$ and $a^{-1} \in C^\delta(R)$.

The following properties follow from the definition of a derivation.

Proposition 12. *If $\delta : R \to R$ is a semiring derivation and $a_1, a_2, \ldots, a_n \in R$ then*

$$\delta\left(\sum_{i=1}^{n} a_i\right) = \sum_{i=1}^{n} \delta\left(a_i\right).$$

$$\delta\left(\prod_{i=1}^{n} a_i\right) = \sum_{k=1}^{n} a_1 \cdot a_2 \cdots a_{k-1} \cdot \delta\left(a_k\right) \cdot a_{k+1} \cdots a_n.$$

Proof. 1 is obvious. Prove 2 by induction on n. The case $n = 2$ is obvious. Suppose the equality holds for $n-1$. Then for n we have

$$\delta\left(\prod_{i=1}^{n} a_i\right) = \delta\left(\prod_{i=1}^{n-1} a_i\right) \cdot a_n + \left(\prod_{i=1}^{n-1} a_i\right) \cdot \delta\left(a_n\right) =$$

$$= \left(\sum_{k=1}^{n-1} a_1 \cdots a_{k-1} \cdot \delta\left(a_k\right) \cdot a_{k+1} \cdots a_{n-1}\right) \cdot a_n + \left(\prod_{i=1}^{n-1} a_i\right) \cdot \delta\left(a_n\right) =$$

$$= \sum_{k=1}^{n} a_1 \cdots a_{k-1} \cdot \delta\left(a_k\right) \cdot a_{k+1} \cdots a_n.$$

Corollary 2. *If R is a multiplicatively commutative δ-semiring and $a_1, a_2, \ldots, a_n \in R$ then*

$$\delta\left(\prod_{i=1}^{n} a_i\right) = \sum_{k=1}^{n} \delta\left(a_k\right) \cdot \prod_{i \neq k} a_i.$$

Define $\delta^2(a) = \delta(\delta(a))$, $\delta^3(a) = \delta(\delta^2(a))$, $\ldots$ By induction we can prove

Proposition 13. *If $\delta \colon R \to R$ is a semiring derivation then $\delta^n \colon R \to R$ is also a semiring derivation for any $n \in \mathbb{N}$.*

Proposition 14. *If $c \in C^{\delta}(R)$ then $\delta^{(n)}(cr) = c\delta^{(n)}(r)$ for any $r \in R$ and $n \in \mathbb{N}$.*

Proof. By induction on n. For $n = 1$ we have $\delta(cr) = \delta(c)r + c\delta(r) = c\delta(r)$. The rest is obvious.

Theorem 1. *If (R, δ) is a differential semiring then for any $a, b \in R$ and $n \in \mathbb{N}$ it holds*

$$\delta^n(ab) = \sum_{k=0}^{n} C_n^k \delta^{n-k}(a)\, \delta^k(b).$$

Proof. By induction on n. The case $n = 1$ is obvious. Suppose the equality holds for $n-1$ and prove it for n.

$$\delta^n(ab) = \delta\left(\sum_{k=0}^{n-1} C_{n-1}^k \delta^{n-k-1}(a)\, \delta^k(b)\right) = \sum_{k=0}^{n-1} C_{n-1}^k \cdot \delta\left(\delta^{n-k-1}(a)\, \delta^k(b)\right) =$$

$$= \sum_{k=0}^{n-1} C_{n-1}^{k} \cdot \left(\delta^{n-k}(a)\,\delta^{k}(b) + \delta^{n-k-1}(a)\,\delta^{k+1}(b)\right) =$$

$$= \delta^{n}(a) \cdot b + \delta^{n-1}(a) \cdot \delta(b) + (n-1)\,\delta^{n-1}(a) \cdot \delta(b) + (n-1)\,\delta^{n-2}(a) \cdot \delta^{2}(b) +$$

$$+ \ldots + C_{n-1}^{k-1} \cdot \delta^{n-k+1}(a) \cdot \delta^{k-1}(b) + C_{n-1}^{k-1} \cdot \delta^{n-k}(a) \cdot \delta^{k}(b) +$$

$$+ C_{n-1}^{k} \cdot \delta^{n-k}(a) \cdot \delta^{k}(b) + C_{n-1}^{k} \cdot \delta^{n-k-1}(a) \cdot \delta^{k+1}(b) + \ldots + \delta(a) \cdot \delta^{n-1}(b) + a \cdot \delta^{n}(b) =$$

$$= \sum_{k=0}^{n} C_{n}^{k} \delta^{n-k}(a)\,\delta^{k}(b).$$

3. Semimodule of semiring derivations. In what follows let R be a additively commutative semiring with absorbing zero. Denote the set of all derivations on R by $\operatorname{Der} R$. In this case $\operatorname{Der} R \neq \varnothing$. Let $\delta_1\colon R \to R$ and $\delta_2\colon R \to R$ be derivations. *The sum of derivations* δ_1 and δ_1 on R is a map $\delta_1 + \delta_2\colon R \to R$ defined by $(\delta_1 + \delta_2)(r) =_{\mathrm{df}} \delta_1(r) + \delta_2(r)$ for all $r \in R$.

In [3] it is shown that the sum of two arbitrary semiring derivations is a semiring derivation. The set $\operatorname{Der} R$ forms a monoid under addition where the neutral element is a trivial derivation. This implies, by induction

Lemma 1. *Let $\delta_i\colon R \to R$ be semiring derivations, $i = 1, 2, \ldots n$. Then $\sum_{i=1}^{n} \delta_i$ is a derivation on R.*

Lemma 2. *Let $\delta\colon R \to R$ be a derivation on R, $a \in Z(R)$. Then the map $a\delta\colon R \to R$ given by $(a\delta)(r) =_{\mathrm{df}} a \cdot \delta(x)$ for all $r \in R$ is a derivation on R.*

Proof. It is trivally checked that $a\delta$ is additive. Let $r, s \in R$ be arbitrary elements. We have $(a\delta)(r \cdot s) = (a\delta(r))\,s + (ar)\,\delta(s) = (a\delta)(r) \cdot s + r\,(a\delta)(s)$.

Corollary 3. *Let R be a semiring, $\delta_i\colon R \to R$ a derivation on R, $\lambda_i \in Z(R)$, $i = 1, 2, ..., n$. Then the map $\sum_{i=1}^{n} \lambda_i \delta_i\colon R \to R$ is a derivation on R.*

Corollary 4. *If $\delta\colon R \to R$ is a derivation of a semiring, then $\alpha_0\delta^n + \alpha_{n-1}\delta^{n-1} + \ldots + \alpha_{n-1}\delta + \alpha_n\varepsilon$ for all $\alpha_i \in R$, is a semiring derivation.*

Theorem 2. *The set $M = \operatorname{Der} R$ of derivations on a semiring R is a semimodule over the center $Z(R)$ of the semiring R.*

Proof. The associativity and commutativity of addition on $\operatorname{Der} R$ follows from the associativity and commutativity of addition on R respectively. The trivial derivation is clearly a neutral element under addition on $\operatorname{Der} R$.

All the other conditions are checked trivially. Let $a \in R$, $\lambda_1, \lambda_2 \in Z(R)$, and $\delta \in Der(R)$ be abritrary elements. Then $((\lambda_1\lambda_2)\,\delta)(a) = (\lambda_1\lambda_2) \cdot \delta(a) = \lambda_1(\lambda_2 \cdot \delta(a)) = \lambda_1((\lambda_2\delta)(a)) = (\lambda_1(\lambda_2\delta))(a)$ follows associativity $((\lambda_1\lambda_2)\delta) = (\lambda_1(\lambda_2\delta))$. Similarly check distributivity $((\lambda_1 + \lambda_2)\,\delta)(a) = (\lambda_1 + \lambda_2) \cdot \delta(a) = \lambda_1 \cdot \delta(a) + \lambda_2 \cdot \delta(a) = (\lambda_1\delta)(a) + (\lambda_2\delta)(a)$, so $((\lambda_1 + \lambda_2)\delta) = (\lambda_1\delta) + (\lambda_2\delta)$.

Let $a \in R$, $\lambda \in Z(R)$, and $\delta_1, \delta_2 \in Der(R)$ be abritrary elements. Then $(\lambda(\delta_1 + \delta_2))(a) = \lambda \cdot (\delta_1 + \delta_2)(a) = \lambda \cdot (\delta_1(a) + \delta_2(a)) = \lambda \cdot \delta_1(a) + \lambda \cdot \delta_2(a) = (\lambda\delta_1)(a) + (\lambda\delta_2)(a)$ follows $\lambda(\delta_1 + \delta_2) = (\lambda\delta_1) + (\lambda\delta_2)$.

Let $a \in R$ and $\delta \in Der(R)$ be abritrary elements. Then $(1 \cdot \delta)(a) = 1 \cdot \delta(a) = \delta(a)$ follows $1_R \cdot \delta = \delta$.

Let $a \in R$, $\lambda \in Z(R)$ and $\delta \in Der(R)$ be abritrary elements. Then $(\lambda \cdot 0_M)(a) = \lambda \cdot 0_M(a) = \lambda \cdot 0_R = 0_R = 0_M(a)$, similarly $(0_R \cdot \delta)(a) = 0_R \cdot \delta(a) = 0_R = 0_M(a)$. It follows $\lambda \cdot 0_M = 0_M$ and $0_R \cdot \delta = 0_M$.

Hence, $Der(R)$ is a $Z(R)$-semimodule.

Corollary 5. *The set $Der(R)$ of derivations of a multiplicatively commutative semiring R is a R-semimodule.*

Theorem 3. *Let $\delta_1 \colon R \to R$, $\delta_2 \colon R \to R$ be derivations of an additively cancellative semiring R. Then $\delta_1\delta_2$ is a derivation on R if and only of $\delta_2(r) \cdot \delta_1(s) + \delta_1(r) \cdot \delta_2(s) = 0$ for all $r, s \in R$.*

Proof. ($\Longrightarrow$) Let $r, s \in R$ be arbitrary elements. Since $\delta_1\delta_2$ is a derivation we have $(\delta_1\delta_2)(rs) = (\delta_1\delta_2)(r) \cdot s + r \cdot (\delta_1\delta_2)(s)$. On the other hand $(\delta_1\delta_2)(rs) = \delta_1(\delta_2(rs)) = \delta_1(\delta_2(r) \cdot s + r \cdot \delta_2(s)) = (\delta_1\delta_2)(r) \cdot s + \delta_2(r) \cdot \delta_1(s) + \delta_1(r) \cdot \delta_2(s) + r \cdot (\delta_1\delta_2)(s)$. It follows

$$(\delta_1\delta_2)(r) \cdot s + r \cdot (\delta_1\delta_2)(s) = (\delta_1\delta_2)(r) \cdot s + \delta_2(r) \cdot \delta_1(s) + \delta_1(r) \cdot \delta_2(s) + r \cdot (\delta_1\delta_2)(s).$$

From additive cancellativeness of R it follows the equality to be proved, i. e.

$$\delta_2(r) \cdot \delta_1(s) + \delta_1(r) \cdot \delta_2(s) = 0.$$

($\Longleftarrow$) Let $r, s \in R$ be arbitrary elements. Then additivity of $\delta_1\delta_2$ follows from the additivity of δ_1 and δ_2 $(\delta_1 \cdot \delta_2)(r+s) = \delta_1(\delta_2(r) + \delta_2(s)) = (\delta_1 \cdot \delta_2)(r) + (\delta_1 \cdot \delta_2)(s)$.

If $\delta_2(r) \cdot \delta_1(s) + \delta_1(r) \cdot \delta_2(s) = 0$ then $(\delta_1\delta_2)(rs) = \delta_1(\delta_2(r) \cdot s + r \cdot \delta_2(s)) = (\delta_1\delta_2)(r) \cdot s + \delta_2(r) \cdot \delta_1(s) + \delta_1(r) \cdot \delta_2(s) + r \cdot (\delta_1\delta_2)(s) = (\delta_1\delta_2)(r) \cdot s + r \cdot (\delta_1\delta_2)(s)$.

Hence $\delta_1\delta_2$ is a derivation on R.

Corollary 6. *Let $\delta_1 \colon R \to R$, $\delta_2 \colon R \to R$ be derivations of an additively commutative semiring R. Then $\delta_1\delta_2$ is a derivation on R if and only if $\delta_2\delta_1$ is a derivation on R.*

Denote by $\{\delta_1, \delta_2\}$ the *anti-commutator* of the derivations δ_1 and δ_2 on R, i. e. $\{\delta_1, \delta_2\} = \delta_1 \cdot \delta_2 + \delta_2 \cdot \delta_1$.

Corollary 7. *If $\delta_1\delta_2$ is a derivation on R then $\{\delta_1, \delta_2\}$ is a derivation on R.*

Theorem 4. *If R is a 2-torsion-free semiring, $\delta : R \to R$ is a derivation, $\{\delta(r), \delta(s)\} = \{r, s\}$ for all $r, s \in R$, then there exists $a \in Z(R)$ such that $a^2 = 1$.*

Proof. From $\{\delta(r), \delta(s)\} = \{r, s\}$ we obtain that $\delta(r) \cdot \delta(s) + \delta(s) \cdot \delta(r) = rs + sr$ for all $r, s \in R$. In particular, $\delta(1) \cdot \delta(1) + \delta(1) \cdot \delta(1) = 1 \cdot 1 + 1 \cdot 1$, which implies $2(\delta(1))^2 = 2$. Since R is 2-torsion-free, then $(\delta(1))^2 = 1$. Hence, $a^2 = 1$, $a \in Z(R)$, where $a = \delta(1)$.

Example. Let R be any semiring, $S \subseteq M_3(R)$ be a subsemiring of $M_3(R)$

$$S = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \middle| \, a, b \in R \right\}$$

under ordinary matrix addition and multiplication. The maps $\delta_1 : S \to S$ and $\delta_2 : S \to S$ defined by the following rules are both derivations on S.

$$\delta_1\left(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \delta_2\left(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that the product of δ_1 and δ_2 is also a derivation on S. Moreover, these derivations commute, i. e. $\delta_1\delta_2 = \delta_2\delta_1$.

Let $V(M)$ be a set of all elements of the semimodule M over a semiring R having additive inverse. The set $V(M)$ is a subsemimodule of the R-semimodule M. []

Denote by $[\delta_1, \delta_2]$ the *commutator* of the derivations δ_1 and δ_2, i. e. a map $[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1$.

Theorem 5. *For all* $\delta_1, \delta_2 \in V(Der(R))$, $[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1 \in V(Der(R))$.

Proof. Let $a, b \in R$. Consider $[\delta_1, \delta_2](a+b) = (\delta_1\delta_2 - \delta_2\delta_1)(a+b) =$
$= \delta_1(\delta_2(a+b)) - \delta_2(\delta_1(a+b)) = \delta_1(\delta_2(a)) + \delta_1(\delta_2(b)) - \delta_2(\delta_1(a)) - \delta_2(\delta_1(b)) =$
$((\delta_1\delta_2)(a) - (\delta_2\delta_1)(a)) + ((\delta_1\delta_2)(b) - (\delta_2\delta_1)(b)) = (\delta_1\delta_2 - \delta_2\delta_1)(a) +$
$+ (\delta_1\delta_2 - \delta_2\delta_1)(b) = [\delta_1, \delta_2](a) + [\delta_1, \delta_2](b)$.

Check the second condition $[\delta_1, \delta_2](ab) = (\delta_1\delta_2 - \delta_2\delta_1)(ab) = \delta_1(\delta_2(ab)) -$
$-\delta_2(\delta_1(ab)) = \delta_1(\delta_2(a)b + a\delta_2(b)) - \delta_2(\delta_1(a)b + a\delta_1(b)) = (\delta_1\delta_2)(a)b +$
$+\delta_2(a)\delta_1(b) + \delta_1(a)\delta_2(b) + a(\delta_1\delta_2)(b) - (\delta_2\delta_1)(a)b - \delta_1(a)\delta_2(b) - \delta_2(a)\delta_1(b) -$
$-a(\delta_2\delta_1)(b) = ((\delta_1\delta_2)(a) - (\delta_2\delta_1)(a))b + a((\delta_1\delta_2)(b) - (\delta_2\delta_1)(b)) =$
$= (\delta_1\delta_2 - \delta_2\delta_1)(a)b + a(\delta_1\delta_2 - \delta_2\delta_1)(b) = [\delta_1, \delta_2](a)b + a[\delta_1, \delta_2](b)$.

4. Conclusions and prospects for further research. In this article we study derivations of semirings and their sets. Namely, we give new examples of such derivations, prove some of their properties. We also prove that the set of all derivations on a semiring forms a semimodule over its center. The obtained results can be used in further research in differential algebra, in a study semirings and semimodules with derivations.

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Мельник І. О., Андрушко А. І. Напівмодуль диференціювань напівкільця.

Вивчаються диференціювання напівкілець, диференціальні напівкільця та множина диференціювань напівкільця. Поняття диференціального напівкільця традиційно означають як адитивне відображення, що задовольняє правило Лейбніца. У статті наведено нові приклади диференціювань напівкілець, доведено деякі їх властивості. Також доведено, що множина всіх диференціювань напівкільця утворює напівмодуль над своїм центром. Показано, що комутатор будь-яких двох диференціювань міститься в піднапівмодулі $V(M)$ елементів M, які мають адитивні обернені.

Ключові слова: диференціювання напівкільця, напівмодуль, напівкільце, піднапівмодуль, диференціальне напівкільце.

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