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STOCHASTICITY AND DETERMINICITY OF MECHANICAL SYSTEMS

The article is devoted to stochasticity and determinicity in mechanical systems: it is shown how a mechanical system can behave in a random way which contradicts the determinicity principle of classical mechanics. A mechanical system is considered, a dumbbell which after being tossed moves in the vertical plane and bounces off the horizontal plane until it stops, one of its tips pointing to the positive direction. On the one hand, the initial conditions uniquely define the dumbbell orientation on the axis, on the other hand, the experiment shows that this is not true. The stochastic nature of the dumbbell behavior is caused by the boundedness from below of measurement device errors and locally "lined" structure of the final state diagram.

Keywords: The determinicity principle, stochasticity, mechanical system, diagram of final states, probability.

1. Introduction. According to the principle of determinicity of classical mechanics the initial position of a mechanical system uniquely defines its trajectory. But experience shows that there exist mechanical systems which movement can not be predicted and these are not unpleasant exceptions but rather a natural situation. One of such simple systems is a dumbbell which after tossing moves in the vertical plane and afterwards bounces off a horizontal plane until it stops with one of its ends pointing to the positive direction. The initial conditions — i.e., the angular velocity and the height from which the dumbbell was thrown, according to the principle of determinicity, uniquely define the orientation of the dumbbell on the abscissa axis. But this is wrong — experience shows that it is impossible to predict (determine) the dumbbell's orientation by the initial conditions. How can it happen?

In this article we undertook an attempt to explain the seeming contradiction between the determinicity principle of classical mechanics and stochastic behavior of a certain mechanical system.

Stochastic behavior of mechanical systems which should behave deterministically according to the determinicity principle of classical mechanics was studied by many authors.

Diaconis, Holmes and Montgomery studied tossing of a coin using a tossing machine (see [1]). The analysis of their model was very careful and comprehensive.

Keller, in particular, considered a mathematical model of flipping a symmetric coin with random initial conditions (see [2]). He supposed that the surface is an absorbing barrier. Keller also considered a wheel and other chance devices.

Nagler and Richter (see [3, 4]) researched dice tossing (a dice was modeled as a barbell with point masses at the tips). The first of these articles is devoted to the symmetrical case of equal masses and the second one contains a generalization of these results to the asymmetric case.

Vulović and Prange studied in [5] a two-dimensional coin toss. They reached a conclusion that the coin flip is not random but closeness to randomness in various randomizers is connected to precision of a tossing mechanism.

2. Model of the dumbbell's movement. The dumbbell is tossed in the vertical plane with the angular velocity ω_0 from height h_0. The dumbbell stops after several bounces off the horizontal plane, one of its tips — white (A) or black (B) — pointing to the positive direction.

The dumbbell's motion is modeled (described) by a system of ordinary differential equations. And the determinicity principle of classical mechanics corresponds to the uniqueness theorem for solutions of an ordinary differential equation.

Let us build a model of the planar motion of the dumbbell in the uniform gravitational field in the vertical plane which passes through the dumbbell's center of mass.

Our dumbbell is modeled as two point masses (the masses of both tips are equal) at the ends of a zero-mass rod. The surface off which the dumbbell bounces is horizontal, absolutely flat and ideally smooth.

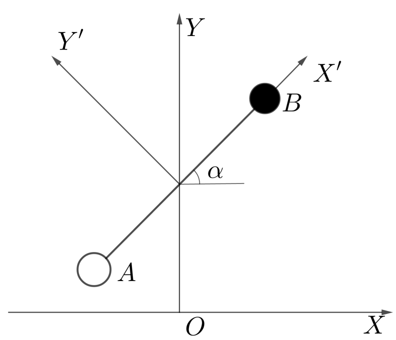

Figure 1. Two Cartesian coordinate systems.

Let us introduce two right-handed Cartesian coordinate systems (see Fig. 1). The first one is an inertial coordinate system Oxy, the Ox axis belongs to the horizontal plane off which the dumbbell bounces, the Oy axis points vertically upwards. The dumbbell's center of mass belongs to the Oy axis. The second coordinate system $O'x'y'$ is a body-fixed coordinate system which is attached to the dumbbell's center of mass, $O'x'$ is pointed along the dumbbell's rod, the positive direction of $O'x'$ is the direction towards the black (B) dumbbell's tip, $O'y'$ is orthogonal to $O'x'$. The orientation of the $O'x'y'$ coordinate system with respect to the Oxy coordinate system is defined by the angle α between Ox and $O'x'$ (see Fig. 1), the positive direction for α is the counterclockwise one.

The dumbbell is a perfectly rigid body. The dumbbell's mass is concentrated at its tips — $m/2$ at each one.

Denote the dumbbell's length by $2l$, the center of mass coordinates by (x, y), the tips' coordinates by (x_A, y_A) and (x_B, y_B) correspondingly, the dumbbell's axial moment of inertia with respect to the $O'z'$ axis by I ($O'z'$ is perpendicular to the plane $O'x'y'$), the dumbbell's angular velocity with respect to the $O'z'$ axis by ω, the acceleration of free fall by g ($g = 9.8$ m/s^2). We neglect the air drag. The friction coefficient is zero (the bounce surface is absolutely smooth). The dumbbell's free motion until the bounce with the surface under the assumptions above is described by the following system of ordinary differential equations:

$$\begin{cases} m\ddot{x} = 0, \\ m\ddot{y} = -gm, \\ I\dot{\omega} = 0, \\ \dot{\alpha} = \omega. \end{cases} \tag{1}$$

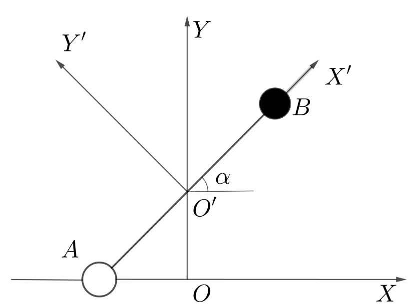

Figure 2. The dumbbell at a bounce moment.

The solution of system (1) until moment t_1 of the first dumbbell's bounce (on $[0; t_1)$ interval) under the initial conditions $x = x_0 = 0$, $\dot{x} = \dot{x}_0 = 0$, $y = y_0 = h$, $\dot{y} = \dot{y}_0 = 0$, $\alpha = \alpha_0$, $\omega = \omega_0$ is

$$\begin{cases} x = 0, \\ y = h + \dot{y}_0 t - \dfrac{gt^2}{2}, \\ \omega = \omega_0, \\ \alpha = \alpha_0 + \omega_0 t. \end{cases} \tag{2}$$

Let us note that $x(t) = 0$ for all t: since the friction coefficient is zero, dumbbell's bounces do not change the center of mass velocity along the Ox axis.

At a moment of a dumbbell's bounce off the surface t_1 (at this moment the ordinate $y_A(t_1)$ of the A tip or the ordinate $y_B(t_1)$ of the B tip is equal to 0) the center of mass velocity and the dumbbell's angular velocity instantly change and the dumbbell continues its movement until the moment t_2 of the second bounce (on $[t_1; t_2)$ interval) according to (1) with the initial conditions

$$x(t_1) = 0,\ \dot{x}(t_1) = 0,\ y(t_1),\ \dot{y}(t_1),\ \alpha(t_1),\ \omega(t_1). \tag{3}$$

The velocity of the dumbbell's center of mass $\dot{y}(t_1)$ and the dumbbell's angular velocity $\omega(t_1)$ are reevaluated in the following way:

$$\dot{y}(t_1) = v_y^{(1)} = \left(1 - \frac{\rho^2\,(1+k)}{r_x^2 + \rho^2}\right) v_y^{(0)} - \frac{(1+k)\, r_x \rho^2}{r_x^2 + \rho^2}\omega^{(0)}, \tag{4}$$

$$\omega(t_1) = \omega^{(1)} = \left(1 - \frac{r_x^2\,(1+k)}{r_x^2 + \rho^2}\right) \omega^{(0)} - \frac{(1+k)\, r_x}{r_x^2 + \rho^2} v_y^{(0)}, \tag{5}$$

where the following notations are used: $\omega^{(0)}$ is the angular velocity before the bounce, $\omega^{(1)}$ is the angular velocity after the bounce, $v_y^{(0)}$ is the projection of the dumbbells' center of mass velocity vector $v^{(0)}$ before the bounce, $v_y^{(1)}$ is the projection of the dumbbells' center of mass velocity vector $v^{(1)}$ after the bounce, $r_x = l\cos\alpha$ if the dumbbell bounces off the surface on its A tip and $r_x = -l\cos\alpha$ if the dumbbell bounces off the surface on its B tip, k is the coefficient of restitution which describes the physical properties of the impact surface ($k = v_y^{(1)}/v_y^{(0)}$), $\rho^2 = I/m = l^2$.

We can find moment t_2 of the second bounce using the known solution of system (1) between the moment t_1 of the first bounce and the moment of the second bounce (on $[t_1, t_2)$ interval).

Then, reevaluating the initial conditions at the moment t_2 of the second bounce (similar to reevaluation of initial conditions at the moment t_1 according to (3)–(5)) we obtain the initial conditions for solving system (1) of differential equations which describes the dumbbell's movement between the moments of the second and the third bounce and so on until the dumbbell stops at some moment (i.e. until it positions itself on the Ox axis or stops in a vertical position) due to lack of energy for further movement. The dumbbell's tip which points to the positive direction of Ox (the white or the black one) is registered, the vertical dumbbell's position is also registered.

The dumbbell's energy at moment t is

$$E(t) = \frac{1}{2}v^2(t)m + \frac{1}{2}I\omega^2(t) + mgy(t).$$

It changes (decreases) at bounce moments t_k (energy is dissipated as a result of a bounce). The dumbbell will continue moving after the k-th bounce (i.e. it will overturn at least once) if its energy $E(t_k)$ after the bounce is not less than $E_{min} = mgl$, i.e. the condition of continuation of the dumbbell's movement after the k-th bounce is the inequality

$$E(t_k) \geq E_{min}. \tag{6}$$

3. The moment of the dumbbell's bounce off the surface. It is necessary to know the moment t_1 of the first dumbbell's bounce off the surface in order to reevaluate the initial conditions at the moment of the bounce according to (3)–(5). We will find it using the known solution (2) of system (1) until the moment t_1 (from now on $\alpha_0 = 0$, $\dot{y}_0 = 0$).

Obviously a moment t of a dumbbell's bounce off the surface must satisfy the equality

$$|l\sin\alpha(t)| = y(t) \tag{7}$$

(see Fig. 2). And the first bounce moment is the smallest root of equation

$$|l\sin(\omega_0 t)| = h - \frac{gt^2}{2}.$$

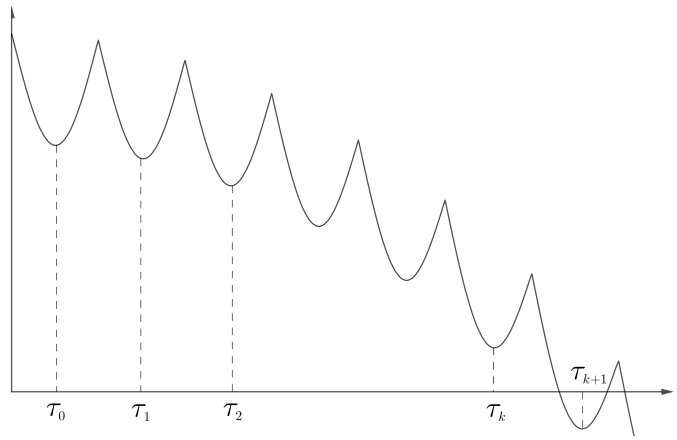

Figure 3. The plot of $f(t)$.

We will find it as follows. The plot of function

$$f(t) = h - \frac{gt^2}{2} - |l \sin(\omega_0 t)|$$

is shown in Fig. 3 (the axis of abscissas corresponds to t, the axis of ordinates — to values $f(t)$). Function $f(t)$ has local minima at the points

$$\tau_k = \frac{1}{\omega_0}\left(\frac{\pi}{2} + k\pi\right), \quad k = 0, 1, 2, \ldots$$

The moment t_1 of the first bounce belongs to the interval $[\tau_k, \tau_{k+1}]$ for which $f(\tau_k) \geq 0$ and $f(\tau_{k+1}) < 0$, besides, t_1 is the only root of equation $f(t) = 0$ from this interval. For given $\varepsilon > 0$ we will find an interval $[\tau_\varepsilon^{(1)}, \tau_\varepsilon^{(2)}] \subset [\tau_k, \tau_{k+1}]$ such that $\tau_\varepsilon^{(2)} - \tau_\varepsilon^{(1)} \leq \varepsilon$, $f(\tau_\varepsilon^{(1)}) > 0$ and $f(\tau_\varepsilon^{(2)}) < 0$. A point t_1 from $[\tau_\varepsilon^{(1)}, \tau_\varepsilon^{(2)}]$ taken randomly according to the uniform distribution on $[\tau_\varepsilon^{(1)}, \tau_\varepsilon^{(2)}]$ will be considered as the moment of the first bounce which is calculated with error ε. Let us note that it is impossible to avoid error when we determine a moment of bounce. We can reevaluate the initial conditions using the known value of moment t_1 of the first bounce and if the energy is big enough for the dumbbell's overturning we can find the moment t_2 of the second bounce, etc. (all the bounce moments are random variables).

We will also consider experiments with an absorbing barrier at the n-th bounce besides the experiment which was described above. An experiment with an absorbing barrier at the n-th bounce will mean the experiment with tossing of a dumbbell and bounces off the surface afterwards such that the n-th bounce is perfectly inelastic and therefore the last one. We register the dumbbell's tip which points to the positive direction of Ox axis.

4. Final state diagram.

We will use the following values for numerical experiments: $2l = 0.6$ m, $m = 2$ kg, $k = 0.8$, $x_0 = 0$ m, $\dot{x}_0 = 0$ m/s, $\dot{y}_0 = 0$ m/s, $\alpha_0 = 0$ rad.

Let us determine the outcome of an experiment for given initial conditions $(\omega, h) \in (-\infty, +\infty) \times [0, +\infty)$: the dumbbell's black tip points towards the positive direction, the white tip points towards the positive direction, the dumbbell stopped vertically. Let us paint the point (ω, h) black if the dumbbell's black tip points towards the positive direction, paint it gray if the white tip points towards the positive direction and paint it white if the dumbbell stopped vertically. We will use the term "final state diagram" for the set of possible initial conditions (ω, h) painted in such a way. Examples of final states diagrams are given in Fig. 4, 5, 6 (these are the final states diagrams for the $[33; 35] \times [81; 83]$ neighborhood of the point $(34; 82)$). The final states diagram for the first bounce (with an absorbing barrier at the first bounce) is shown in Fig. 4, the final states diagrams for the second and the third bounces are shown in Fig. 5 and 6 correspondingly. The final states diagrams for the subsequent bounces have structure similar to the structure shown in Fig. 6.

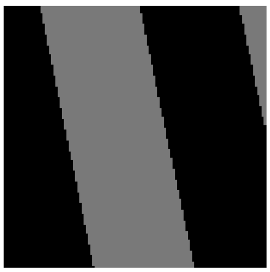

Figure 4. Diagram, first bounce.

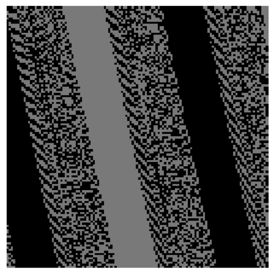

Figure 5. Diagram, second bounce.

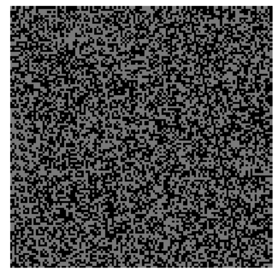

Figure 6. Diagram, third bounce.

5. Local structure of a final state diagram. We will give examples of local structure of final state diagrams for the case when the dumbbell starts movement with high initial energy — the dumbbell has high initial angular velocity and and it is tossed from large height, or at least one of this parameters has a large value.

Enlarged final state diagrams for the $[85-10^{-4}; 85+10^{-4}]\times[0.5-10^{-4}; 0.5+10^{-4}]$ neighborhood of the point $(85; 0.5)$ for the 4th, 5th and 6th bounces are shown in Fig. 7, 8 and 9 correspondingly. We see "uniformly gray" structure of the final state diagrams but really the final state diagrams have locally line structure (see Fig. 10, 11, 12). The enlarged fragment $[85 - 10^{-7}; 85 + 10^{-7}] \times [0.5 - 3 \cdot 10^{-7}; 0.5 + 3 \cdot 10^{-7}]$ of a neighborhood of the point $(85; 0.5)$ at the 4th bounce is shown in Fig. 10. The enlarged fragment $[85-10^{-9}; 85+10^{-9}]\times[0.5-3\cdot10^{-9}; 0.5+3\cdot10^{-9}]$ of a neighborhood of the point $(85; 0.5)$ at the 5th bounce is shown in Fig. 11. The enlarged fragment $[85 - 10^{-11}; 85 + 10^{-11}] \times [0.5 - 3 \cdot 10^{-11}; 0.5 + 3 \cdot 10^{-11}]$ of a neighborhood of the point $(85; 0.5)$ at the 6th bounce is shown in Fig. 12. (Let us note that the hydrogen atom size is 10^{-11} m.)

The final state diagram for the case when the contribution of h into the dumbbell's energy is substantially greater than the the contribution of ω has locally "vertical" line structure.

The final state diagram will also have locally line structure for other quite large values of the dumbbell's energy and the slope of stripes (see Fig. 4) depends on contribution of ω and h into the dumbbell's energy.

6. Measurement errors and local structure of a final state diagram. Locally line structure of final state diagram holds for large values of the dumbbell's energy for a numerical experiment. This fact provides grounds for concluding that final state diagram for a physical experiment also has locally line structure when energy is large.

We set the initial conditions (the initial angular velocity ω_0 and the initial height h_0) using a measurement device. Boundedness of the measurement error Δ of the measurement device from below — it cannot be less than certain Δ_0 (the error cannot be arbitrarily small) and locally line structure of the final state diagram (when energy is large enough) make impossible prediction of the experiment outcome (i.e. the dumbbell's orientation) from the initial conditions. By "setting" the initial conditions (ω_0, h_0) using a measurement device which has the measurement error $\Delta = (\Delta_\omega, \Delta_h)$ we actually obtain as initial conditions a point (ω, h) from a rectangle $[\omega_0 - \Delta_\omega/2, \omega_0 + \Delta_\omega/2] \times [h_0 - \Delta_h/2, h_0 + \Delta_h/2]$. A measurement device which has the measurement error $\Delta = (\Delta_\omega, \Delta_h)$ cannot distinguish between initial conditions (ω_1, h_1) and (ω_2, h_2) such that $|\omega_1 - \omega_2| < \Delta_\omega$ and $|h_1 - h_2| < \Delta_h$. But a mechanical system "distinguishes" between arbitrarily close but different initial conditions. Therefore every time when we set the initial conditions using a measurement device we actually obtain as initial conditions a random point (ω, h) from a neighborhood $[\omega_0 - \Delta_\omega/2, \omega_0 + \Delta_\omega/2] \times [h_0 - \Delta_h/2, h_0 + \Delta_h/2]$. But the final state diagram for the $[\omega_0 - \Delta_\omega/2, \omega_0 + \Delta_\omega/2] \times [h_0 - \Delta_h/2, h_0 + \Delta_h/2]$ neighborhood of (ω_0, h_0) point has line structure. This fact makes impossible determination of the dumbbell's orientation using the initial conditions (ω_0, h_0).

When we conduct the numerical experiment the boundedness of error Δ for the initial conditions (ω_0, h_0) from below is modeled as random choice of a point (ω, h) from the $[\omega_0 - \Delta_\omega/2, \omega_0 + \Delta_\omega/2] \times [h_0 - \Delta_h/2, h_0 + \Delta_h/2]$ neighborhood of (ω_0, h_0) point. The second error (which is impossible to avoid during a numerical simulation of the dumbbell's movement) is the error of determining a bounce moment, this error depends on the accuracy of a computer and a calculations algorithm. But the system (1) "can distinguish" between arbitrarily close but different initial conditions. The line structure of the final state diagram for the $[\omega_0 - \Delta_\omega/2, \omega_0 + \Delta_\omega/2] \times [h_0 - \Delta_h/2, h_0 + \Delta_h/2]$ neighborhood of (ω_0, h_0) and boundedness of the measurement device error from below (this error is greater or equal to a certain Δ_0) make impossible determination of the experiment's outcome by the initial conditions (ω_0, h_0) in a numerical experiment.

Figure 7. Enlarged, fourth bounce.

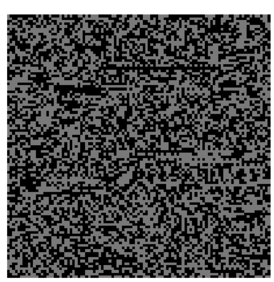

Figure 8. Enlarged, fifth bounce.

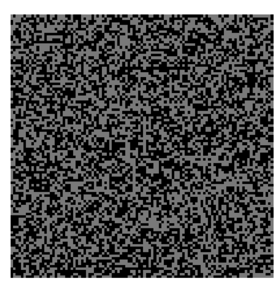

Figure 9. Enlarged, sixth bounce.

The results we obtain do not contradict either the principle of determinicity of classical mechanics or the uniqueness theorem for solutions of an ordinary differential equation. The phenomenon of non-deterministic, stochastic dumbbell's behavior is caused by fundamental boundedness of measurement device error from below and locally line structure of the final state diagram.

The numerical simulation of the dumbbell's movement was realized by V.I. Dubyna, Master student of Oles Honchar Dnipro National University.

Figure 10.
Line structure,
fourth bounce.

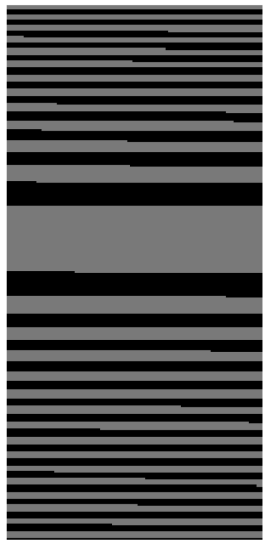

Figure 11.
Line structure,
fifth bounce.

Figure 12.
Line structure,
sixth bounce.

7. Conclusions. Initial conditions uniquely define behavior of a mechanical system by reason of the principle of determinicity of classical mechanics. A mechanical system is considered in the article which behaves stochastically, it is shown why the principle of determinicity does not hold for this system.

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Турчин В. М., Пироженко О. В. Стохастичніть і детермінованість механічних систем.

Робота присвячена стохастичності і детермінованості у механічних системах. Згідно з принципом детермінованості класичної механіки початкові умови однозначно визначають еволюцію системи у часі. Проте досвід свідчить, що це далеко не завжди так.

Розглядається механічна система — гантель після підкидання рухається у вертикальній площині співударяючсь з горизонтальною площиною, допоки рано чи пізно не зупиниться тим чи іншим кінцем у додатньому напрямі. Яким саме — передбачити неможливо, що на перший погляд протирічить принципу детермінованості. Насправді протиріччя немає — стохастичний характер поведіки гантелі обумовлений принциповою обмеженістю знизу похибки вимірювального приладу і локальною "лінійчастою" структурою діаграми кінцевих станів.

Ключові слова: Принцип детермінованості, стохастичність, механічна система, діаграма кінцевих станів, імовірність.

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