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# INTERPOLATION PROBLEM FOR PERIODICALLY CORRELATED PROCESSES WITH MISSING OBSERVATIONS

The problem of the optimal linear estimation of a linear functional depending on the unknown values of periodically correlated stochastic process from observations of the process with additive noise with missing observations is considered. Formulas for calculating the mean square error and the spectral characteristic of the optimal linear estimate of the functional are proposed in the case where spectral densities are exactly known. Formulas that determine the least favorable spectral densities and the minimax spectral characteristics are proposed in the case of spectral uncertainty, where the spectral densities are not exactly known while some classes of admissible spectral densities are specified.

**Keywords:** periodically correlated stochastic process, spectral characteristics, mean-square error, minimax (robust) estimate, least favorable spectral density, minimax spectral characteristics.

**1. Introduction.** The investigation of cyclostationarity as a phenomenon was started by W. R. Bennett in 1958, [1]. He studied the statistical characteristics of signals in channels of communication and called the group of such signals the cyclostationary process. That is a nonstationary process having a periodically varying in time statistical characteristics. Literature review of theory and application of cyclostationarity in different spheres of research is presented in the article [2] by W. A. Gardner, A. Napolitano, L. Paura. In other sources cyclostationary processes are called periodically stationary, periodically nonstationary, periodically correlated. We will use the term periodically correlated processes.

E. G. Gladyshev, [3], was one of the first who started the study of periodically correlated processes with continuous time. H. L. Hurd continued study of periodically correlated processes and their properties in the papers [4], [5]. A. Makagon in [6] investigated relations between periodically correlated processes and stationary processes.

A. N. Kolmogorov [7], N. Wiener [8], A. M. Yaglom [9], [10], proposed their methods of solution of estimation problems for stationary processes and sequences in the case of spectral certainty. In the case where complete information on the spectral densities is impossible, but a set of admissible spectral densities is given, the minimax approach to estimation problem is used. That is we find estimate that minimizes the mean square error for all spectral densities from a given class of densities simultaneously. U. Grenander, [11], was the first who applied the minimax estimation method to find solution of the extrapolation problem for stationary processes.

The detailed analysis of the estimation problems with missing observations are presented in the books by M. J. Daniels and J. W. Hogan [12], P. E. McKnight et al [13]. The interpolation and extrapolation problems of linear functionals from periodically correlated stochastic sequences with missing observations were investigated by I. I. Golichenko and M. P. Moklyachuk in [14], [15], by I. I. Golichenko, O. Yu. Masyutka and M. P. Moklyachuk in [16].

In this paper we study the problem of mean square optimal linear estimation of the functional $A_s\zeta = \sum_{l=0}^{s-1}\int_{M_l}^{M_l+N_{l+1}} a(t)\zeta(t)dt$ which depends on the unknown values of a periodically correlated stochastic process $\zeta(t)$. The estimation is based on observations of the process $\zeta(t)+\theta(t)$ at points $t\in\mathbb{R}\setminus S$, $S=\bigcup_{l=0}^{s-1}[M_l, M_l+N_{l+1}]$, $M_l=\sum_{k=0}^{l}(N_k+K_k)$, $N_0=K_0=0$. We obtain formulas for calculation the mean square error and the spectral characteristic of the optimal linear estimate of $A_s\zeta$ in the case of spectral certainty. The least favorable spectral density and the minimax (robust) spectral characteristic of the optimal linear estimate of $A_s\zeta$ are found in the case when the spectral density is not known, but the class of admissible densities is given.

## 2. Periodically correlated processes and generated vector stationary sequences.

**Definition 1.** *[3] Mean square continuous stochastic process* $\zeta:\mathbb{R}\to H=L_2(\Omega,\mathcal{F},\mathbb{P})$, $\mathsf{E}\zeta(t)=0$, *is called periodically correlated (PC) with period* $T$, *if its correlation function* $K(t+u,u)=\mathsf{E}\zeta(t+u)\overline{\zeta(u)}$ *for all* $t,u\in\mathbb{R}$ *and some fixed* $T>0$ *is such that*

$$K(t+u,u)=K(t+u+T,u+T).$$

Let $\{\zeta(t), t\in\mathbb{R}\}$ and $\{\theta(t), t\in\mathbb{R}\}$ be mutually uncorrelated PC processes. We construct two sequences of stochastic functions

$$\{\zeta_j(u)=\zeta(u+jT), u\in[0,T), j\in\mathbb{Z}\}, \tag{1}$$

$$\{\theta_j(u)=\theta(u+jT), u\in[0,T), j\in\mathbb{Z}\}. \tag{2}$$

Sequences (1) and (2) form the $L_2([0,T);H)$-valued stationary sequences $\{\zeta_j, j\in\mathbb{Z}\}$ and $\{\theta_j, j\in\mathbb{Z}\}$, respectively, with the correlation functions

$$B_\zeta(l,j)=\langle\zeta_l,\zeta_j\rangle_H=\int_0^T K_\zeta(u+(l-j)T,u)du=B_\zeta(l-j),$$

$$B_\theta(l,j)=\langle\theta_l,\theta_j\rangle_H=\int_0^T K_\theta(u+(l-j)T,u)du=B_\theta(l-j),$$

where $K_\zeta(t,s)=\mathsf{E}\zeta(t)\overline{\zeta(s)}$, $K_\theta(t,s)=\mathsf{E}\theta(t)\overline{\theta(s)}$ are correlation functions of PC processes $\zeta(t)$, $\theta(t)$.

Let us define in $L_2([0,T);\mathbb{R})$ the orthonormal basis

$$\{\widetilde{e}_k=\frac{1}{\sqrt{T}}e^{2\pi i\{(-1)^k\left[\frac{k}{2}\right]\}u/T}, k=1,2,\dots\},\ \langle\widetilde{e}_j,\widetilde{e}_k\rangle=\delta_{kj}.$$

Stationary sequences $\{\zeta_j, j\in\mathbb{Z}\}$ and $\{\theta_j, j\in\mathbb{Z}\}$ can be represented in the form

$$\zeta_j=\sum_{k=1}^{\infty}\zeta_{kj}\widetilde{e}_k,\ \zeta_{kj}=\langle\zeta_j,\widetilde{e}_k\rangle=\frac{1}{\sqrt{T}}\int_0^T\zeta_j(v)e^{-2\pi i\{(-1)^k\left[\frac{k}{2}\right]\}v/T}dv, \tag{3}$$

$$\theta_j=\sum_{k=1}^{\infty}\theta_{kj}\widetilde{e}_k,\quad \theta_{kj}=\langle\theta_j,\widetilde{e}_k\rangle. \tag{4}$$

Let us name sequences

$$\{\zeta_j, j\in\mathbb{Z}\},\ \{\theta_j, j\in\mathbb{Z}\},$$

or corresponding vector sequences

$$\{\vec{\zeta_j}=(\zeta_{kj},k=1,2,...)^{\top}, j\in\mathbb{Z}\},\ \{\vec{\theta_j}=(\theta_{kj},k=1,2,...)^{\top}, j\in\mathbb{Z}\}$$

generated vector stationary sequences, that means that vector sequences $\{\vec{\zeta_j}=(\zeta_{kj},k=1,2,...)^{\top}, j\in\mathbb{Z}\}$, $\{\vec{\theta_j}=(\theta_{kj},k=1,2,...)^{\top}, j\in\mathbb{Z}\}$ are generated by processes $\{\zeta(t),t\in\mathbb{R}\},\{\theta(t),t\in\mathbb{R}\}$, respectively.

Components $\{\zeta_{kj}, k=1,2,\dots\}$ and $\{\theta_{kj}, k=1,2,\dots\}$ of stationary sequences $\{\zeta_j, j\in\mathbb{Z}\}$ and $\{\theta_j, j\in\mathbb{Z}\}$ are such that, [17],

$$\mathsf{E}\zeta_{kj}=0,\ \ \|\zeta_j\|_H^2=\sum_{k=1}^{\infty}\mathsf{E}|\zeta_{kj}|^2=P_\zeta=B_\zeta(0)<\infty,\ \ \mathsf{E}\zeta_{kl}\overline{\zeta_{nj}}=\langle R_\zeta(l-j)e_k,e_n\rangle,$$

$$\mathsf{E}\theta_{kj}=0,\ \ \|\theta_j\|_H^2=\sum_{k=1}^{\infty}\mathsf{E}|\theta_{kj}|^2=P_\theta=B_\theta(0)<\infty,\ \ \mathsf{E}\theta_{kl}\overline{\theta_{nj}}=\langle R_\theta(l-j)e_k,e_n\rangle.$$

where $\{e_k, k=1,2,\dots\}$ is a basis of the space $\ell_2$. Correlation functions $R_\zeta(j)$ and $R_\theta(j)$ of generated vector stationary sequences $\{\zeta_j, j\in\mathbb{Z}\}$ and $\{\theta_j, j\in\mathbb{Z}\}$ are correlation operator functions in $\ell_2$. Correlation operators $R_\zeta(0)=R_\zeta$, $R_\theta(0)=R_\theta$ are kernel operators and their kernel norms satisfy the following restrictions:

$$\sum_{k=1}^{\infty}\langle R_\zeta e_k,e_k\rangle=\|\zeta_j\|_H^2=P_\zeta,\ \sum_{k=1}^{\infty}\langle R_\theta e_k,e_k\rangle=\|\theta_j\|_H^2=P_\theta,$$

Generated vector stationary sequences $\{\zeta_j, j\in\mathbb{Z}\}$, $\{\theta_j, j\in\mathbb{Z}\}$ have spectral density functions $f(\lambda)=\{f_{kn}(\lambda)\}_{k,n=1}^{\infty}$, $g(\lambda)=\{g_{kn}(\lambda)\}_{k,n=1}^{\infty}$ that are positive operator valued functions of variable $\lambda\in[-\pi,\pi)$ in $\ell_2$, if their correlation functions $R_\zeta(j)$ and $R_\theta(j)$ can be represented in the form

$$\langle R_\zeta(j)e_k,e_n\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ij\lambda}\langle f(\lambda)e_k,e_n\rangle d\lambda,$$

$$\langle R_\theta(j)e_k,e_n\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ij\lambda}\langle g(\lambda)e_k,e_n\rangle d\lambda,\ k,n=1,2,\dots.$$

Spectral densities $f(\lambda)$, $g(\lambda)$ a.e. on $[-\pi,\pi)$ are kernel operators with integrable kernel norms:

$$\sum_{k=1}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\langle f(\lambda)e_k,e_k\rangle d\lambda=\|\zeta_j\|_H^2=P_\zeta,\ \sum_{k=1}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\langle g(\lambda)e_k,e_k\rangle d\lambda=\|\theta_j\|_H^2=P_\theta.$$

**Hilbert space projection method of linear interpolation.**

Let consider the problem of mean square optimal linear estimation of the functional

$$A_s\zeta = \sum_{l=0}^{s-1} \int\limits_{M_l}^{M_l+N_{l+1}} a(t)\zeta(t)dt,$$

which depends on the unknown values of a periodically correlated stochastic process $\zeta(t)$. The estimation is based on observations of the process $\zeta(t)+\theta(t)$ at points $t \in \mathbb{R}\setminus S$, $S = \bigcup_{l=0}^{s-1}[M_l, M_l+N_{l+1}]$, $M_l = \sum_{k=0}^{l}(N_k+K_k)$, $N_0 = K_0 = 0$. Process $\theta(t)$ is uncorrelated with $\zeta(t)$ periodically correlated process. The function $a(t), t \in \mathbb{R}$, satisfies the condition $\sum_{l=0}^{s-1}\int_{M_l}^{M_l+N_{l+1}} |a(t)|dt < \infty$.

We assume that length of each interval of observations is a multiple of the period $T$ and the length of each interval of missed observations is a multiple of $T$, what means that

$$K_1 = T\cdot K_1^T, K_2 = T\cdot K_2^T, \ldots, K_{s-1} = T\cdot K_{s-1}^T,$$
$$N_1 = T\cdot N_1^T, N_2 = T\cdot N_2^T, \ldots, N_s = T\cdot N_s^T,$$

respectively. The set which corresponds to the set $S$ is of the form

$$\widetilde{S} = \bigcup_{l=0}^{s-1}\left\{M_l^T, ..., M_l^T + N_{l+1}^T - 1\right\}.$$

The functional $A_s\zeta$ can be written as

$$A_s\zeta = \sum_{l=0}^{s-1}\int_{M_l}^{M_l+N_{l+1}} a(t)\zeta(t)dt = \sum_{l=0}^{s-1}\sum_{j=M_l^T}^{M_l^T+N_{l+1}^T-1}\int_0^T a(u+jT)\zeta(u+jT)du,$$

where $M_l = T\cdot M_l^T, l = 0, ..., s-1$.

Denoting by $a(u+jT) = a_j(u)$, $\zeta(u+jT) = \zeta_j(u)$, $j \in \widetilde{S}$, $u \in [0,T)$, and taking into account the decomposition (3) of generated vector stationary sequence $\{\zeta_j, j\in\mathbb{Z}\}$, [18], the functional $A_s\zeta$ can be written as

$$A_s\zeta = \sum_{l=0}^{s-1}\sum_{j=M_l^T}^{M_l^T+N_{l+1}^T-1}\int_0^T a(u+jT)\zeta(u+jT)du =$$

$$= \sum_{l=0}^{s-1}\sum_{j=M_l^T}^{M_l^T+N_{l+1}^T-1}\left(a_{1j}\zeta_{1j} + \sum_{n=2}^{\infty} a_{n+(-1)^n,j}\,\zeta_{nj}\right) = \sum_{l=0}^{s-1}\sum_{j=M_l^T}^{M_l^T+N_{l+1}^T-1}\vec{a}_j^{\top}\vec{\zeta}_j,$$

where vectors $\vec{a}_j$ have the special form

$$\vec{a}_j = (a_{kj}, k=1,2,\ldots)^{\top} = (a_{1j}, a_{3j}, a_{2j}, \ldots, a_{2k+1,j}, a_{2k,j}, \ldots)^{\top},\ j\in\tilde{S},$$

where $a_{kj} = \langle a_j, \widetilde{e}_k\rangle = \frac{1}{\sqrt{T}}\int_0^T a_j(v)e^{-2\pi i\{(-1)^k\left[\frac{k}{2}\right]\}v/T}dv$, $\vec{\zeta}_j = (\zeta_{kj}, k=1,2,\ldots)^{\top}$, $j\in\widetilde{S}$, is generated vector stationary sequence.

Let us assume that coefficients $\{\vec{a}_j, j \in \widetilde{S}\}$ satisfy conditions

$$\|\vec{a}_j\| < \infty, \quad \|\vec{a}_j\|^2 = \sum_{k=1}^{\infty} |a_{kj}|^2, \quad j \in \tilde{S}. \tag{5}$$

On condition (5) functional $A_s\zeta$ has finite second moment.

Let the spectral densities $f^{\zeta}(\lambda)$ and $f^{\theta}(\lambda)$ satisfy the minimality condition

$$\int_{-\pi}^{\pi} Tr\big[(f^{\zeta}(\lambda) + f^{\theta}(\lambda))^{-1}\big] d\lambda < +\infty. \tag{6}$$

Condition (6) is necessary and sufficient in order that the error-free interpolation of the unknown values of the sequence $\vec{\zeta}_j + \vec{\theta}_j$ is impossible, [19].

Denote by $L_2(f)$ the Hilbert space of vector valued functions $\vec{b}(\lambda) = \{b_\nu(\lambda)\}_{\nu=1}^{\infty}$ that are square integrable with respect to a measure with the density $f(\lambda) = \{f_{\nu\mu}(\lambda)\}_{\nu,\mu=1}^{\infty}$: $\int_{-\pi}^{\pi} \vec{b}^{\top}(\lambda) f(\lambda) \overline{\vec{b}(\lambda)} d\lambda = \int_{-\pi}^{\pi} \sum_{\nu,\mu=1}^{\infty} b_\nu(\lambda) f_{\nu\mu}(\lambda) \overline{b_\mu(\lambda)} d\lambda < +\infty$.

Denote by $L_2^s(f)$ the subspace in $L_2(f)$ generated by the functions $e^{ij\lambda}\delta_\nu, \delta_\nu = \{\delta_{\nu\mu}\}_{\mu=1}^{\infty}, j \in \mathbb{Z} \setminus \widetilde{S},\ \nu = 1, \dots,$ where $\delta_{\nu\nu} = 1, \delta_{\nu\mu} = 0$ for $\nu \neq \mu$.

Every mean-square optimal linear estimate $\widehat{A_s\zeta}$ of the functional $A_s\zeta$ from observations of the sequence $\zeta_j + \theta_j$ at points $j \in \mathbb{Z} \setminus \widetilde{S}$ has the form

$$\widehat{A_s\zeta} = \int_{-\pi}^{\pi} \vec{h}^{\top}(e^{i\lambda})(Z^{\zeta}(d\lambda) + Z^{\theta}(d\lambda)) = \int_{-\pi}^{\pi} \sum_{\nu=1}^{\infty} h_\nu(e^{i\lambda})(Z_\nu^{\zeta}(d\lambda) + Z_\nu^{\theta}(d\lambda)), \tag{7}$$

where $Z^{\zeta}(\Delta) = \left\{Z_\nu^{\zeta}(\Delta)\right\}_{\nu=1}^{\infty}$ and $Z^{\theta}(\Delta) = \left\{Z_\nu^{\theta}(\Delta)\right\}_{\nu=1}^{\infty}$ are orthogonal random measures of the sequences $\zeta_j$ and $\theta_j$, and $\vec{h}(e^{i\lambda}) = \left\{h_\nu(e^{i\lambda})\right\}_{\nu=1}^{\infty}$ is the spectral characteristic of the estimate $\widehat{A_s\zeta}$. The function $\vec{h}(e^{i\lambda}) \in L_2^s(f^{\zeta} + f^{\theta})$.

The mean square error $\Delta(\vec{h}; f^{\zeta}, f^{\theta})$ of the estimate $\widehat{A_s\zeta}$ is calculated by the formula

$$\Delta(\vec{h}; f^{\zeta}, f^{\theta}) = E|A_s\zeta - \widehat{A_s\zeta}|^2 =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[A_s(e^{i\lambda}) - \vec{h}(e^{i\lambda})\right]^{\top} f^{\zeta}(\lambda) \overline{\left[A_s(e^{i\lambda}) - \vec{h}(e^{i\lambda})\right]} d\lambda + \frac{1}{2\pi} \int_{-\pi}^{\pi} \vec{h}^{\top}(e^{i\lambda}) f^{\theta}(\lambda) \overline{\vec{h}(e^{i\lambda})} d\lambda, \tag{8}$$

where $A_s(e^{i\lambda}) = \sum_{l=0}^{s-1} \sum_{j=M_l^T}^{M_l^T + N_{l+1}^T - 1} \vec{a}_j e^{ij\lambda}$.

The spectral characteristic $\vec{h}(f^{\zeta}, f^{\theta})$ of the optimal linear estimate of $A_s\zeta$ minimizes the mean square error

$$\Delta(f^{\zeta}, f^{\theta}) = \Delta(\vec{h}(f^{\zeta}, f^{\theta}); f^{\zeta}, f^{\theta}) = \min_{\vec{h} \in L_2^s(f^{\zeta} + f^{\theta})} \Delta(\vec{h}; f^{\zeta}, f^{\theta}) = \min_{\widehat{A_s\zeta}} \mathsf{E}|A_s\zeta - \widehat{A_s\zeta}|^2. \tag{9}$$

Optimal estimate $\widehat{A_s\zeta}$ is a solution of optimization problem (9). With the help of the Hilbert space projection method proposed by A. N. Kolmogorov, [7], we can find a solution of the optimization problem (9). The optimal linear estimate $\widehat{A_s\zeta}$ is a projection of the functional $A_s\zeta$ on the subspace $H^-[\zeta + \theta] = H^-[\zeta_{kj} + \theta_{kj},\ j \in$

$\mathbb{Z}\backslash\widetilde{S}, k=1,2,\dots]$ of the Hilbert space $H=\{\xi : \mathsf{E}\xi=0,\ \mathsf{E}|\xi|^2<\infty\}$. The projection is characterized by the following conditions: 1) $\widehat{A_s\zeta}\in H^-[\zeta+\theta]$, 2) $A_s\zeta-\widehat{A_s\zeta}\perp H^-[\zeta+\theta]$.

From condition 2) it follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(A_s^\top(e^{i\lambda})f^\zeta(\lambda)-h^\top(e^{i\lambda})(f^\zeta(\lambda)+f^\theta(\lambda))\right)e^{-ij\lambda}d\lambda=0,\ j\in\mathbb{Z}\setminus\widetilde{S}. \tag{10}$$

From (10) we can derive the formula for the spectral characteristic of the optimal estimate

$$\vec{h}^\top(f^\zeta,f^\theta)=\left(A_s^\top(e^{i\lambda})f^\zeta(\lambda)-C_s^\top(e^{i\lambda})\right)\left[f^\zeta(\lambda)+f^\theta(\lambda)\right]^{-1}=$$
$$=A_s^\top(e^{i\lambda})-\left(A_s^\top(e^{i\lambda})f^\theta(\lambda)+C_s^\top(e^{i\lambda})\right)\left[f^\zeta(\lambda)+f^\theta(\lambda)\right]^{-1}, \tag{11}$$

where $C_s(e^{i\lambda})=\sum_{l=0}^{s-1}\sum_{k_l=M_l^T}^{M_l^T+N_{l+1}^T-1}\vec{c}_{k_l}e^{ik_l\lambda}$, column-vectors of unknown coefficients $\vec{c}_{k_l}=(c_{kk_l},\,k\geq 1)=(c_{1k_l},c_{2k_l},\dots)^\top$, $l=0,\dots,s-1$, $k_l=M_l^T,\dots,M_l^T+N_{l+1}^T-1$.

Condition 1) is satisfied when the system of equalities

$$\int_{-\pi}^{\pi}\vec{h}^\top(f^\zeta,f^\theta)e^{-ij\lambda}d\lambda=0,\quad j\in\widetilde{S}, \tag{12}$$

holds true.

Let us define operators $\mathbf{D}_s$, $\mathbf{B}_s$ that are determined by matrices

$$\mathbf{D}_s=\begin{pmatrix} D_{00} & D_{01} & \dots & D_{0,s-1}\\ D_{10} & D_{11} & \dots & D_{1,s-1}\\ \dots & \dots & \dots & \dots\\ D_{s-1,0} & D_{s-1,1} & \dots & D_{s-1,s-1}\end{pmatrix},\quad \mathbf{B}_s=\begin{pmatrix} B_{00} & B_{01} & \dots & B_{0,s-1}\\ B_{10} & B_{11} & \dots & B_{1,s-1}\\ \dots & \dots & \dots & \dots\\ B_{s-1,0} & B_{s-1,1} & \dots & B_{s-1,s-1}\end{pmatrix},$$

constructed from block-matrices

$$D_{mn}=\{D_{mn}(k,j)\}_{k=M_m^T}^{M_m^T+N_{m+1}^T-1}{}_{j=M_n^T}^{M_n^T+N_{n+1}^T-1},\ B_{mn}=\{B_{mn}(k,j)\}_{k=M_m^T}^{M_m^T+N_{m+1}^T-1}{}_{j=M_n^T}^{M_n^T+N_{n+1}^T-1},$$
$$m,n=0,\dots,s-1,$$

with elements

$$D_{mn}(k,j)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f^\zeta(\lambda)(f^\zeta(\lambda)+f^\theta(\lambda))^{-1}\right]^\top e^{i(j-k)\lambda}d\lambda,$$

$$B_{mn}(k,j)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[(f^\zeta(\lambda)+f^\theta(\lambda))^{-1}\right]^\top e^{i(j-k)\lambda}d\lambda,$$

$$k=M_m^T,\dots,M_m^T+N_{m+1}^T-1,\ j=M_n^T,\dots,M_n^T+N_{n+1}^T-1,\ m,n=0,\dots,s-1.$$

With the help of the defined operators, relation (12) can be written in the form of the equation

$$\mathbf{D}_s\vec{a}_s=\mathbf{B}_s\vec{c}_s, \tag{13}$$

where $\vec{a}_s$, $\vec{c}_s$ are column-vectors

$$\vec{a}_s=\left(\vec{a}_0^\top,\dots,\vec{a}_{N_1^T-1}^\top,\vec{a}_{M_1^T}^\top\dots,\vec{a}_{M_1^T+N_2^T-1}^\top,\dots,\vec{a}_{M_{s-1}^T}^\top,\dots,\vec{a}_{M_{s-1}^T+N_s^T-1}^\top\right)^\top,$$

$$\vec{c}_s=\left(\vec{c}_0^{\top},\dots,\vec{c}_{N_1^T-1}^{\top},\vec{c}_{M_1^T}^{\top}\dots,\vec{c}_{M_1^T+N_2^T-1}^{\top},\dots,\vec{c}_{M_{s-1}^T}^{\top},\dots,\vec{c}_{M_{s-1}^T+N_s^T-1}^{\top}\right)^{\top}.$$

If the inverse matrix for the block-matrix $\mathbf{B}_s$ exists, the unknown components $\vec{c}_{k_l}$, $l=0,\dots,s-1$, $k_l=M_l^T,\dots,M_l^T+N_{l+1}^T-1$, of vector $\vec{c}_s$ are determined from the equation (13).

The mean-square error of the optimal estimate $\widehat{A_s\zeta}$ is calculated by the formula (8) and is of the form

$$\Delta(f^{\zeta},f^{\theta})=\langle\vec{a}_s,\mathbf{R}_s\vec{a}_s\rangle+\langle\vec{c}_s,\mathbf{B}_s\vec{c}_s\rangle, \tag{14}$$

where $\langle a,b\rangle$ denotes the scalar product, $\mathbf{R}_s$ is the linear operator determined by matrix $\mathbf{R}_s=\{R_{mn}\}_{m.n=0}^{s-1}$ composed with block-matrices

$$R_{mn}=\{R_{mn}(k,j)\}_{k=M_m^T}^{M_m^T+N_{m+1}^T-1}\,{}_{j=M_n^T}^{M_n^T+N_{n+1}^T-1},\ m,n=0,\dots,s-1,$$

with elements

$$R_{mn}(k,j)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f^{\zeta}(\lambda)(f^{\zeta}(\lambda)+f^{\theta}(\lambda))^{-1}f^{\theta}(\lambda)\right]^{\top}e^{i(j-k)\lambda}d\lambda,$$

$$k=M_m^T,\dots,M_m^T+N_{m+1}^T-1,\ j=M_n^T+1,\dots,M_n^T+N_{n+1}^T-1.$$

The following statement holds true.

**Theorem 1.** *Let $\{\zeta(t),t\in\mathbb{R}\}$ and $\{\theta(t),t\in\mathbb{R}\}$ be mutually uncorrelated PC processes such that stationary sequences $\{\zeta_j,j\in\mathbb{Z}\}$ and $\{\theta_j,j\in\mathbb{Z}\}$, which are built by relations (1), (2), respectively, have spectral density matrices $f^{\zeta}(\lambda)$ and $f^{\theta}(\lambda)$. Assume that the matrices $f^{\zeta}(\lambda)$ and $f^{\theta}(\lambda)$ satisfy the minimality condition (6). Let coefficients $\{\vec{a}_j,j=0,1,\dots\}$ that determine the functional $A_s\zeta$ satisfy conditions (5). Then the spectral characteristic $\vec{h}(f^{\zeta},f^{\theta})$ and the mean square error $\Delta(f^{\zeta},f^{\theta})$ of the optimal estimate of the functional $A_s\zeta$ from observations of the process $\zeta(t)+\theta(t)$ at points $t\in\mathbb{R}\setminus S$ are given by (11), (14). The optimal estimate $\widehat{A_s\zeta}$ of the functional $A_s\zeta$ is calculated by the formula (7).*

In the case of observations without noise we have the following corollary.

**Corollary 1.** *Let $\{\zeta(t),t\in\mathbb{R}\}$ be a PC process such that stationary sequence $\{\zeta_j,j\in\mathbb{Z}\}$, which is built by relations (1), has spectral density matrix $f^{\zeta}(\lambda)$. Assume that the matrix $f^{\zeta}(\lambda)$ satisfies the minimality condition*

$$\int_{-\pi}^{\pi}Tr\left[(f^{\zeta}(\lambda))^{-1}\right]d\lambda<+\infty. \tag{15}$$

*Let coefficients $\{\vec{a}_j,j=0,1,\dots\}$ that determine the functional $A_s\zeta$ satisfy conditions (5). The spectral characteristic $\vec{h}(f^{\zeta})$ and the mean square error $\Delta(f^{\zeta})$ of the optimal linear estimate of the functional $A_s\zeta$ based on observations of the process $\zeta(t)$ at points $t\in\mathbb{R}\setminus S$, are calculated by formulas*

$$\vec{h}^{\top}(f^{\zeta})=A_s^{\top}(e^{i\lambda})-C_s^{\top}(e^{i\lambda})\left[f^{\zeta}(\lambda)\right]^{-1}, \tag{16}$$

$$\Delta(f^{\zeta})=\langle\vec{c}_s,\vec{a}_s\rangle, \tag{17}$$

*where* $\vec{a}_s$, $\vec{c}_s$ *are column-vectors, components of vector* $\vec{c}_s$ *are determined from the equation* $\boldsymbol{B}_s\vec{c}_s = \vec{a}_s$, *if the inverse matrix for* $\boldsymbol{B}_s$ *exists. Operator* $\boldsymbol{B}_s$ *is constructed from block-matrices* $B_{mn} = \{B_{mn}(k,j)\}_{k=M_m^T}^{M_m^T+N_{m+1}^T-1}{}_{j=M_n^T}^{M_n^T+N_{n+1}^T-1}$, $m,n = 0,\dots,s-1$, *with elements which are the Fourier coefficients of the matrix function* $\left[(f^{\zeta}(\lambda))^{-1}\right]^{\top}$*:*

$$B_{mn}(k,j) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[(f^{\zeta}(\lambda))^{-1}\right]^{\top} e^{i(j-k)\lambda}d\lambda,$$

$$k = M_m^T,\dots,M_m^T+N_{m+1}^T-1,\; j = M_n^T,\dots,M_n^T+N_{n+1}^T-1,\; m,n = 0,\dots,s-1.$$

**Minimax (robust) method of linear interpolation.**

Let $f(\lambda)$ and $g(\lambda)$ be the spectral density matrices of generated stationary sequences $\zeta_j$ and $\theta_j$, which are built by relations (1), (2), respectively.

Formulas (11), (14) and (16), (17) may be applied for finding the spectral characteristic and the mean square error of the optimal linear estimate of the functional $A_s\zeta$ only under the condition that the spectral density matrices $f(\lambda)$ and $g(\lambda)$ are exactly known. But when the density matrices are not known exactly while a set $D = D_f \times D_g$ of admissible spectral densities is given, the minimax (robust) approach to estimation of functionals from unknown values of stationary sequences is used. In this case we find the estimate which minimizes the mean square error for all spectral densities from the given set simultaneously.

**Definition 2.** *For a given class of pairs of spectral densities* $D = D_f \times D_g$ *the spectral density matrices* $f^0(\lambda) \in D_f$, $g^0(\lambda) \in D_g$ *are called the least favorable in* $D$ *for the optimal linear estimation of the functional* $A_s\zeta$ *if*

$$\Delta(f^0,g^0) = \Delta(\vec{h}(f^0,g^0);f^0,g^0) = \max_{(f,g)\in D}\Delta(\vec{h}(f,g);f,g).$$

**Definition 3.** *For a given class of pairs of spectral densities* $D = D_f \times D_g$ *the spectral characteristic* $\vec{h}^0(\lambda)$ *of the optimal linear estimate of the functional* $A_s\zeta$ *is called minimax (robust) if*

$$\vec{h}^0(\lambda) \in H_D = \bigcap_{(f,g)\in D} L_2^s(f+g),$$

$$\min_{\vec{h}\in H_D}\max_{(f,g)\in D}\Delta(\vec{h};f,g) = \max_{(f,g)\in D}\Delta(\vec{h}^0;f,g).$$

Taking into consideration these definitions and the obtained relations we can verify that the following lemmas hold true.

**Lemma 1.** *The spectral density matrices* $f^0(\lambda) \in D_f$, $g^0(\lambda) \in D_g$, *that satisfy condition (6), are the least favorable in* $D$ *for the optimal linear estimation of* $A_s\zeta$, *if the Fourier coefficients of the matrix functions* $(f^0(\lambda)+g^0(\lambda))^{-1}$, $f^0(\lambda)(f^0(\lambda)+g^0(\lambda))^{-1}$, $f^0(\lambda)(f^0(\lambda)+g^0(\lambda))^{-1}g^0(\lambda)$ *define matrices* $\boldsymbol{B}_s{}^0, \boldsymbol{D}_s{}^0, \boldsymbol{R}_s{}^0$, *that determine a solution of the constrained optimization problem*

$$\max_{(f,g)\in D}\left(\langle\vec{a}_s,\boldsymbol{R}_s\vec{a}_s\rangle + \langle(\boldsymbol{B}_s)^{-1}\boldsymbol{D}_s\vec{a}_s,\boldsymbol{D}_s\vec{a}_s\rangle\right) = \langle\vec{a}_s,\boldsymbol{R}_s{}^0\vec{a}_s\rangle + \langle(\boldsymbol{B}_s{}^0)^{-1}\boldsymbol{D}_s{}^0\vec{a}_s,\boldsymbol{D}_s{}^0\vec{a}_s\rangle.$$

*The minimax spectral characteristic* $\vec{h}^0 = \vec{h}(f^0,g^0)$ *is given by (11), if* $\vec{h}(f^0,g^0) \in H_D$.

**Lemma 2.** *The spectral density matrix $f^0(\lambda) \in D_f$, that satisfies condition (15), is the least favorable in $D_f$ for the optimal linear estimation of $A_s\zeta$ based on observations of the process $\zeta(t)$ at points $t \in \mathbb{R} \setminus S$, if the Fourier coefficients of the matrix function $(f^0(\lambda))^{-1}$ define the matrix $\boldsymbol{B}_s{}^0$, that determine a solution of the constrained optimization problem*

$$\max_{f \in D_f} \langle (\boldsymbol{B}_s)^{-1} \vec{a}_s, \vec{a}_s \rangle = \langle (\boldsymbol{B}_s{}^0)^{-1} \vec{a}_s, \vec{a}_s \rangle.$$

*The minimax spectral characteristic $\vec{h}^0 = \vec{h}(f^0)$ is given by (16), if $\vec{h}(f^0) \in H_D$.*

The least favorable spectral densities $f^0(\lambda) \in D_f$, $g^0(\lambda) \in D_g$ and the minimax spectral characteristic $\vec{h}^0 = \vec{h}(f^0, g^0)$ form a saddle point of the function $\Delta(\vec{h}; f, g)$ on the set $H_D \times D$. The saddle point inequalities

$$\Delta(\vec{h}^0; f, g) \leq \Delta(\vec{h}^0; f^0, g^0) \leq \Delta(\vec{h}; f^0, g^0), \quad \forall \vec{h} \in H_D, \forall f \in D_f, \forall g \in D_g$$

hold when $\vec{h}^0 = \vec{h}(f^0, g^0)$, $\vec{h}(f^0, g^0) \in H_D$ and $(f^0, g^0)$ is a solution of the constrained optimization problem

$$\sup_{(f,g) \in D_f \times D_g} \Delta\left(\vec{h}(f^0, g^0); f, g\right) = \Delta\left(\vec{h}(f^0, g^0); f^0, g^0\right). \tag{18}$$

The linear functional $\Delta(\vec{h}(f^0, g^0); f, g)$ is calculated by the formula

$$\Delta(\vec{h}(f^0, g^0); f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(A_s(e^{i\lambda}) g^0(\lambda) + C_s^0(e^{i\lambda})\right)^{\top} (f^0(\lambda) + g^0(\lambda))^{-1} f(\lambda) \times$$

$$(f^0(\lambda) + g^0(\lambda))^{-1} \overline{(A_s(e^{i\lambda}) g^0(\lambda) + C_s^0(e^{i\lambda}))} d\lambda + \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(A_s(e^{i\lambda}) f^0(\lambda) - C_s^0(e^{i\lambda})\right)^{\top} \times$$

$$(f^0(\lambda) + g^0(\lambda))^{-1} g(\lambda) (f^0(\lambda) + g^0(\lambda))^{-1} \overline{(A_s(e^{i\lambda}) f^0(\lambda) - C_s^0(e^{i\lambda}))} d\lambda.$$

In the case of estimation of the functional based on observations without noise we have the following statement.

**Lemma 3.** *Let $f^0(\lambda)$ satisfies the condition (15) and be a solution of the constrained optimization problem*

$$\Delta(\vec{h}(f^0); f) \to sup, f(\lambda) \in D_f, \tag{19}$$

$$\Delta(\vec{h}(f^0); f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(C_s^0(e^{i\lambda})\right)^{\top} (f^0(\lambda))^{-1} f(\lambda) (f^0(\lambda))^{-1} \overline{(C_s^0(e^{i\lambda}))} d\lambda.$$

*Then $f^0(\lambda)$ is the least favorable spectral density matrix for the optimal linear estimation of $A_s\zeta$ based on observations of the process $\zeta(t)$ at points $t \in \mathbb{R} \setminus S$. The minimax spectral characteristic $\vec{h}^0 = \vec{h}(f^0)$ is given by (16), if $\vec{h}(f^0) \in H_D$.*

**The least favorable spectral densities in $D_0^-$.**

Let $\{\zeta(t), t \in \mathbb{R}\}$ be a PC process such that stationary sequence $\{\zeta_j, j \in \mathbb{Z}\}$, which is built by relations (1), has spectral density matrix $f^{\zeta}(\lambda)$. Assume that the length of each interval of observations is a multiple of the period $T$: $K_1 =$

$T \cdot K_1^T, K_2 = T \cdot K_2^T, \ldots, K_{s-1} = T \cdot K_{s-1}^T$, and the length of each interval of missed observations is a multiple of $T$: $N_1 = T \cdot N_1^T, N_2 = T \cdot N_2^T, \ldots, N_s = T \cdot N_s^T$.

Consider the problem of minimax estimation of the functional $A_s\zeta$ from observations of the process $\zeta(t)$ at points $t \in \mathbb{R} \setminus S$ without noise, under the condition that the spectral density matrix $f(\lambda)$ of generated stationary sequence $\zeta_j$ belongs to the set

$$D_0^- = \left\{ f(\lambda) | \, \frac{1}{2\pi} \int_{-\pi}^{\pi} f^{-1}(\lambda) d\lambda = P \right\},$$

where $P = \{p_{\nu\mu}\}_{\nu,\mu=1}^{\infty}$ is a given positive definite matrix and $detP \neq 0$.

With the help of Lemma 3 and the method of Lagrange multipliers we can find that a solution $f^0(\lambda)$ of the constrained optimization problem (19) satisfy the following relation:

$$\left[(f^0(\lambda))^{-1}\right]^\top C_s^0(e^{i\lambda}) = \left[(f^0(\lambda))^{-1}\right]^\top \vec{\alpha}, \tag{20}$$

where $\vec{\alpha} = \{\alpha_k\}_{k=1}^{\infty}$ – a vector of Lagrange multipliers, $C_s^0(e^{i\lambda}) = \sum_{l=0}^{s-1} \sum_{j=M_l^T}^{M_l^T+N_{l+1}^T-1} \vec{c}_j^{\,0} e^{ij\lambda}$, $\vec{c}_s^{\,0} = \{(\vec{c}_j^{\,0})^\top\}_{j\in\tilde{S}}$ – column-vector of unknown coefficients $\vec{c}_j^{\,0}$, $j \in \tilde{S}$, which are determined from relation $\mathbf{B}_s^0 \vec{c}_s^{\,0} = \vec{a}_s$, the matrix $\mathbf{B}_s^0$ is constructed from the Fourier coefficients of the matrix function $\left[(f^0(\lambda))^{-1}\right]^\top$:

$$B_s^0(k,j) = R^\top(k-j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[(f^0(\lambda))^{-1}\right]^\top e^{i(j-k)\lambda} d\lambda, \ k,j \in \tilde{S},$$

The Fourier coefficients $R(k) = R^*(-k), k \in \tilde{S}$, found from the equation $\mathbf{B}_s^0 \vec{\alpha}_s = \vec{a}_s$, for $\vec{\alpha}_s = (\vec{\alpha}, \vec{0}, \ldots, \vec{0}, \ldots)^\top$, satisfy relation (20) and $\mathbf{B}_s^0 \vec{c}_s^{\,0} = \vec{a}_s$. From equations above we obtain that

$$R(k) = \begin{cases} P(\vec{a}_0)^{-1} \vec{a}_k^\top, \ k \in \tilde{S}, \\ 0, \ k \in \{0, \ldots, M_{s-1}^T + N_s^T - 1\} \backslash \tilde{S}, \end{cases}$$

where $\left[(\vec{a}_0)^{-1}\right]^\top \cdot \vec{a}_0 = 1$. The equality $R(0) = P$ follows as a consequence of the restriction on the spectral densities from the class $D_0^-$.

Let the vector-valued sequence $\vec{a}_k, k \in \tilde{S}$, be such that the matrix function $(f^0(\lambda))^{-1} = \sum_{k=-\left(M_{s-1}^T+N_s^T-1\right)}^{M_{s-1}^T+N_s^T-1} R(k) e^{ik\lambda}$ is positive definite and has nonzero determinant. Then $(f^0(\lambda))^{-1}$ can be represented in the form, [20],

$$(f^0(\lambda))^{-1} = \left( \sum_{k=0}^{M_{s-1}^T+N_s^T-1} Q(k) e^{-ik\lambda} \right) \cdot \left( \sum_{k=0}^{M_{s-1}^T+N_s^T-1} Q(k) e^{-ik\lambda} \right)^*,$$

where $Q(k) = 0$ – zero matrix for $k \in \{0, \ldots, M_{s-1}^T + N_s^T - 1\} \setminus \tilde{S}$. Thus $f^0(\lambda)$ is the spectral density of the vector autoregression stochastic sequence of order $M_{s-1}^T + N_s^T - 1$ generated by the equation

$$\sum_{k=0}^{M_{s-1}^T+N_s^T-1} Q(k) \vec{\zeta}_{n-k} = \vec{\varepsilon}_n, \tag{21}$$

where $\vec{\varepsilon}_n$ is a vector "white noise" sequence. The minimax spectral characteristic $\vec{h}(f^0)$ is given by

$$\vec{h}(f^0) = -\sum_{k=1}^{M_{s-1}^T+N_s^T-1} \overline{R(k)}(P^T)^{-1}\vec{a}_0 e^{-ik\lambda}. \tag{22}$$

Hence the following theorem holds true.

**Theorem 2.** *Let the sequence $\vec{a}_j$, $j \in \widetilde{S}$, which determine the linear functional $A_s\zeta$ from observations of the process $\zeta(t)$ at points $t \in \mathbb{R} \setminus S$ without noise, be such that the matrix function $\sum_{k=-\left(M_{s-1}^T+N_s^T-1\right)}^{M_{s-1}^T+N_s^T-1} R(k)e^{ik\lambda}$, where*

$$R(k) = R^*(-k) = \begin{cases} P(\vec{a}_0)^{-1}\vec{a}_k^\top, \ k \in \tilde{S}, \\ 0, \ k \in \{0, \dots, M_{s-1}^T + N_s^T - 1\}\backslash\tilde{S}, \end{cases}$$

*is positive definite and has nonzero determinant. Then the least favorable in the class $D_0^-$ spectral density for the optimal linear estimate of $A_s\zeta$ is given by the formula*

$$f^0(\lambda) = \left( \sum_{k=-\left(M_{s-1}^T+N_s^T-1\right)}^{M_{s-1}^T+N_s^T-1} R(k)e^{ik\lambda} \right)^{-1}. \tag{23}$$

*The minimax spectral characteristic $\vec{h}(f^0)$ is given by (22).*

**The least favorable spectral densities in $D_M^-$.**

Let $\{\zeta(t), t \in \mathbb{R}\}$ be a PC process such that stationary sequence $\{\zeta_j, j \in \mathbb{Z}\}$, which is built by relations (1), has spectral density matrix $f^\zeta(\lambda)$. Assume that the length of each interval of observations is a multiple of the period $T$: $K_1 = T \cdot K_1^T, K_2 = T \cdot K_2^T, \dots, K_{s-1} = T \cdot K_{s-1}^T$, and the length of each interval of missed observations is a multiple of $T$: $N_1 = T \cdot N_1^T, N_2 = T \cdot N_2^T, \dots, N_s = T \cdot N_s^T$.

Consider the problem of minimax estimation of the functional $A_s\zeta$ from observations of the process $\zeta(t)$ at points $t \in \mathbb{R} \setminus S$ without noise, under the condition that the spectral density matrix $f(\lambda)$ of generated stationary sequence $\zeta_j$ belongs to the set

$$D_0^- = \left\{ f(\lambda) | \, \frac{1}{2\pi} \int_{-\pi}^{\pi} f^{-1}(\lambda) \cos(m\lambda) d\lambda = P(m), \ m = 0, 1, ..., M \right\},$$

where the sequence of matrices $P(m) = \{P_{\nu\mu}(m)\}_{\nu,\mu=1}^\infty$, $P(m) = P^*(-m)$, $m = 0, ..., M$, is such that the matrix function $\sum_{m=-M}^{M} P(m)e^{im\lambda}$ is positive definite and has the determinant that does not equal zero.

With the help of Lemma 3 and the method of Lagrange multipliers we can find that solution $f^0(\lambda)$ of the constrained optimization problem (19) satisfy the following relation:

$$\begin{gathered} \left[(f^0(\lambda))^{-1}\right]^\top C_s{}^0(e^{i\lambda}) \left(C_s{}^0(e^{i\lambda})\right)^* \left[(f^0(\lambda))^{-1}\right]^\top = \\ = \left[(f^0(\lambda))^{-1}\right]^\top \left(\sum_{m=0}^{M} \vec{\alpha}_m e^{im\lambda}\right) \left(\sum_{m=0}^{M} \vec{\alpha}_m e^{im\lambda}\right)^* \left[(f^0(\lambda))^{-1}\right]^\top, \end{gathered} \tag{24}$$

where $\vec{\alpha}_m$, $m=0,1,\dots,M$ are Lagrange multipliers. Relation (24) holds true if

$$\sum_{j\in\tilde{S}} \vec{c}_j^{\,0} e^{ij\lambda} = \sum_{m=0}^{M} \vec{\alpha}_m e^{im\lambda}.$$

Consider two cases: 1) $M \geq M_{s-1}^T + N_s^T - 1$ and 2) $M < M_{s-1}^T + N_s^T - 1$.

Let 1) $M \geq M_{s-1}^T + N_s^T - 1$. Then the Fourier coefficients of the function $(f^0(\lambda)^{-1})^\top$ determine the matrix $\mathbf{B}_s^0$ and $\vec{\alpha}_{M_{s-1}^T+N_s^T} = \vec{\alpha}_{M_{s-1}^T+N_s^T+1} = \dots = \vec{\alpha}_M = \vec{0}$. Thus, extremum problem (19) is degenerate.

Let $\vec{\alpha}_{M_{s-1}^T+N_s^T} = \dots = \vec{\alpha}_M = \vec{0}$ and $\vec{\alpha}_m = 0$, $m \notin \tilde{S}$, and $\vec{\alpha}_0,\dots,\vec{\alpha}_{M_{s-1}^T+N_s^T-1}$ find from the equation $\mathbf{B}_s^0 \vec{\alpha}_s^0 = \vec{a}_s$, where $\vec{\alpha}_s^0 = \left(\vec{\alpha}_0,\dots,\vec{\alpha}_{M_{s-1}^T+N_s^T-1}\right)^\top$. Then the least favorable density satisfies the relation

$$f^0(\lambda) = \left(\sum_{m=-M}^{M} P(m) e^{im\lambda}\right)^{-1} = \left(\left(\sum_{m=0}^{M} Q(m) e^{-im\lambda}\right)\left(\sum_{m=0}^{M} Q(m) e^{-im\lambda}\right)^*\right)^{-1} \tag{25}$$

This spectral density $f^0(\lambda)$ is the density of the vector stochastic autoregression sequence of the order $M$

$$\sum_{m=0}^{M} Q(m)\vec{\zeta}_{l-m} = \vec{\varepsilon}_l. \tag{26}$$

Let 2) $M < M_{s-1}^T + N_s^T - 1$. Then the matrix $\mathbf{B}_s$ is defined by the Fourier coefficients of the function $(f(\lambda)^{-1})^\top$. Among them $P(m), m \in \{0,\dots,M\} \cap \tilde{S}$, are known and $P(m), m \in \tilde{S} \setminus \{0,\dots,M\}$, are unknown. The unknown coefficients $\vec{\alpha}_m$, $m \in \{0,\dots,M\} \cap \tilde{S}$ and $P(m), m \in \tilde{S} \setminus \{0,\dots,M\}$ we find from the equation

$$\mathbf{B}_s \vec{\alpha}_M^0 = \vec{a}_s, \tag{27}$$

where $\tilde{\alpha}_M^0 = (\vec{\alpha}_0,\dots,\vec{\alpha}_{M'},\vec{0},\dots,\vec{0})^\top$, $M'$ is defined from the relation $\{0,\dots,M\} \cap \tilde{S} = \{0,\dots,M'\}$.

The equation (27) can be represented as a system of the following equations $\sum_{m\in\{0,\dots,M\}\cap\tilde{S}} B_s(j,m)\vec{\alpha}_m = \vec{a}_j$, $j \in \tilde{S}$. From the first $M'$ equations we can find coefficients $\vec{\alpha}_0,\dots,\vec{\alpha}_{M'}$ and from the next equations we can find matrices $P(m), m \in \tilde{S} \setminus \{0,\dots,M\}$.

If the sequence of matrices $P(m), m \in \tilde{S}$, is such that $P(m) = P^*(m), m \in \tilde{S}$, the matrix function $\sum_{m=-\left(M_{s-1}^T+N_s^T-1\right)}^{M_{s-1}^T+N_s^T-1} P(m)e^{im\lambda}$ is positive-definite and has the determinant which does not equal zero identically, then the least favorable spectral density $f^0(\lambda)$ is defined by the formula

$$f^0(\lambda) = \left(\sum_{m=-\left(M_{s-1}^T+N_s^T-1\right)}^{M_{s-1}^T+N_s^T-1} P(m)e^{im\lambda}\right)^{-1} \tag{28}$$

and $f^0(\lambda) = \left(\left(\sum_{m=0}^{M_{s-1}^T+N_s^T-1} Q(m)e^{-im\lambda}\right)\left(\sum_{m=0}^{M_{s-1}^T+N_s^T-1} Q(m)e^{-im\lambda}\right)^*\right)^{-1}$. This spectral density $f^0(\lambda)$ is the density of the vector stochastic autoregression sequence of the order $M_{s-1}^T + N_s^T - 1$

$$\sum_{m=0}^{M_{s-1}^T + N_s^T - 1} Q(m)\vec{\zeta}_{l-m} = \vec{\varepsilon}_l. \tag{29}$$

Thus, the following theorem holds true.

**Theorem 3.** *If $M \geq M_{s-1}^T + N_s^T - 1$ then the least favorable spectral density in the class $D_M^-$ for the optimal estimation of the functional $A_s\zeta$ is defined by (25). It is spectral density of the vector stochastic autoregression sequence (26) of order $M$, that is determined by matrices $P(m), m \in \{0, 1, \dots, M\}$.*

*If $M < M_{s-1}^T + N_s^T - 1$ and solutions $P(m), m \in \tilde{S} \cap \{0, 1, \dots, M\}$, of the equation $\mathbf{B}_s\vec{\alpha}_M^0 = \vec{a}_s$ with coefficients $P(m), m \in \tilde{S} \backslash \{0, 1, \dots, M\}$, form a positive-definite matrix function $\sum_{m=-\left(M_{s-1}^T + N_s^T - 1\right)}^{M_{s-1}^T + N_s^T - 1} P(m)e^{im\lambda}$, with the determinant which does not equal zero identically, then the spectral density (28) of the vector stochastic autoregression sequence (29) of order $M_{s-1}^T + N_s^T - 1$ is the least favorable in the class $D_M^-$. The minimax spectral characteristic $h(f^0)$ is calculated by the formula (16).*

**3. Conclusions and prospects for further research.** We propose formulas for calculating the mean square error and the spectral characteristic of the optimal linear estimate of the functional $A_s\zeta = \sum_{l=0}^{s-1} \int_{M_l}^{M_l + N_{l+1}} a(t)\zeta(t)dt$ which depends on the unknown values of a periodically correlated stochastic process $\zeta(t)$. The estimation is based on observations of the process $\zeta(t) + \theta(t)$ at points $t \in \mathbb{R} \setminus S$, $S = \bigcup_{l=0}^{s-1} [M_l, M_l + N_{l+1}]$, $M_l = \sum_{k=0}^{l} (N_k + K_k)$, $N_0 = K_0 = 0$. Process $\theta(t)$ is uncorrelated with $\zeta(t)$ periodically correlated process.

The problem is considered under conditions of spectral certainty and spectral uncertainty. In the first case the spectral density matrices $f^{\zeta}(\lambda)$ and $f^{\theta}(\lambda)$ of the generated vector stationary sequences are known exactly. In this case we derived formulas for calculating the spectral characteristic and the mean-square error of the optimal estimate of the functional. In the second case the spectral density matrices are not exactly known, but a class $D = D_f \times D_g$ of admissible spectral densities is specified. Formulas that determine the least favorable spectral densities and the minimax spectral characteristic of the optimal estimate of the functional $A_s\zeta$ are proposed. The problem is investigated in details for classes $D_0^-$, $D_M^-$ of admissible spectral densities.

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**Голіченко І. І., Моклячук М. П.** Задача інтерполяції для періодично корельованого процесу із пропущеними значеннями.

Досліджується задача оптимального лінійного оцінювання функціонала від невідомих значень періодично корельованого стохастичного процесу за результатами спостережень процесу з адитивним шумом із пропущеними значеннями. У випадку відомих спектральних щільностей виведено формули для обчислення спектральної характеристики та середньоквадратичної похибки оптимальної оцінки функціонала. У випадку спектральної невизначеності, коли спектральні щільності не відомі точно, але визначено клас допустимих спектральних щільностей, запропоновано формули для обчислення найменш сприятливих спектральних щільностей та мінімаксних спектральних характеристик оптимальних лінійних оцінок функціоналів.

**Ключові слова:** періодично корельований процес, спектральна характеристика, середньоквадратична похибка, мінімаксна (робастна) оцінка, найменш сприятлива спектральна щільність, мінімаксні спектральні характеристики.

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