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LOCAL NEARRINGS AND ENDOCYCLIC GROUPS OF SMALL ORDER

Nearrings are generalization of associative rings, in which the additive group can be non-abelian, and addition is connected with multiplication at least one distributive law, left or right. In this sense local nearrings are generalization of local rings.

The package SONATA of the computer algebra system GAP contains a library of all non-isomorphic nearrings of order at most 15 and nearrings with a unity of order at most 31, among which 698 are local. However, the classification of nearrings of higher orders requires much more complex calculations.

The classification of `nearrings with identity` (up to order 32) and `local nearrings` (of orders 32 and 64) on `endocyclic additive groups` is given. Moreover, some information concerning these nearring from the `LocalNR` and `SONATA packages` of `GAP` is presented.

Keywords: nearring with identity, local nearring, endocyclic group, additive group, LocalNR package, GAP.

1. Introduction. We study algebraic structures called nearrings, which are interesting examples of generalised rings (i.e. addition need not to be commutative, and only one distributive law is assumed).

The classification of all nearrings up to certain orders is an open problem. The SONATA package [1] of GAP [2] contains a library of all non-isomorphic nearrings of order at most 15 and nearrings with identity of order up to 31, among which 698 are local. We have implemented algorithms to compute all local nearrings of further orders, in a new GAP package called LocalNR [3]. The current version of this package contains all local nearrings of order at most 361, except some orders.

2. Nearrings with identity and endocyclic groups.

Definition 1. *A set N with two binary operations $+$ and $\cdot$ is called a (left) nearring if the following statements hold:*

1) $(N,+) = N^+$ is a (not necessarily abelian) group with neutral element 0;
2) $(N,\cdot)$ is a semigroup;
3) $x(y+z) = xy + xz$ for all $x, y, z \in N$.

The monographs by Pilz [4], Meldrum [5], Clay [6], C. Ferrero and G. Ferrero [7], and Lockhart [8] are devoted to the general theory of nearrings.

Let G be a group and $\operatorname{End} G$ be the set of all its endomorphisms, which can be considered as a semigroup with respect to the composition operation of endomorphisms. For each $g \in G$ we denote by $g^{\operatorname{End} G}$ the set $\{g^\alpha | \alpha \in \operatorname{End} G\}$ of all images of the element g with respect to endomorphisms of $\operatorname{End} G$.

Definition 2. *A group G is called endocyclic if it contains an element g with $G = g^{\operatorname{End} G}$.*

The SONATA package includes the *AllLibraryNearRingsWithOne* database of nearrings with identity. In this library, nearrings are arranged according to their additive groups G, for example, the command

$$N := AllLibraryNearRingsWithOne(G)$$

defines the lists of local nearrings N.

We use the function $IsEndoCyclicGroup$ from the LocalNR package for checking whether G is an endocyclic group.

Let $[n, i]$ be the i-th group of order n in the SmallGroups library in GAP. We denote by C_n, D_n and Q_n the cyclic, dihedral and quaternion groups of order n, respectively.

Let $m(G)$ be the number of all non-isomorphic nearrings with identity R whose additive group R^+ is isomorphic to the group G.

The SONATA package includes the list of all non-isomorphic nearrings with identity of order less than 32.

The following lemma are obtained using GAP, the LocalNR and SONATA packages.

Lemma 1. *The following groups G of order less than 32 are endocyclic. Moreover, if G is an additive group of a nearring with identity R, then the following holds.*

IdGroup(G)	*StructureDescription(G)*	$m(G)$
$[4,1]$	C_4	*1*
$[4,2]$	$C_2 \times C_2$	*5*
$[5,1]$	C_5	*1*
$[6,2]$	C_6	*1*
$[7,1]$	C_7	*1*
$[8,1]$	C_8	*1*
$[8,2]$	$C_4 \times C_2$	*10*
$[8,3]$	D_8	*7*
$[8,4]$	Q_8	-
$[8,5]$	$C_2 \times C_2 \times C_2$	*35*
$[9,1]$	C_9	*1*
$[9,2]$	$C_3 \times C_3$	*10*
$[10,2]$	C_{10}	*1*
$[11,1]$	C_{11}	*1*
$[12,2]$	C_{12}	*1*

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IdGroup(G)	*StructureDescription(G)*	$m(G)$
$[12, 4]$	D_{12}	*1*
$[12, 5]$	$C_6 \times C_2$	*9*
$[13, 1]$	C_{13}	*1*
$[14, 2]$	C_{14}	*1*
$[15, 1]$	C_{15}	*1*
$[16, 1]$	C_{16}	*1*
$[16, 2]$	$C_4 \times C_4$	*51*
$[16, 3]$	$(C_4 \times C_2) \rtimes C_2$	*132*
$[16, 4]$	$C_4 \rtimes C_4$	*40*
$[16, 5]$	$C_8 \times C_2$	*37*
$[16, 6]$	$C_8 \rtimes C_2$	*33*
$[16, 7]$	D_{16}	-
$[16, 10]$	$C_4 \times C_2 \times C_2$	*470*
$[16, 11]$	$C_2 \times D_8$	*708*
$[16, 12]$	$C_2 \times Q_8$	*4*
$[16, 14]$	$C_2 \times C_2 \times C_2 \times C_2$	*2798*
$[17, 1]$	C_{17}	*1*
$[18, 2]$	C_{18}	*1*
$[18, 3]$	$C_3 \times S_3$	*8*
$[18, 5]$	$C_6 \times C_3$	*17*
$[19, 1]$	C_{19}	*1*
$[20, 2]$	C_{20}	*1*
$[20, 4]$	D_{20}	*1*
$[20, 5]$	$C_{10} \times C_2$	*9*
$[21, 2]$	C_{21}	*1*
$[22, 2]$	C_{22}	*1*
$[23, 1]$	C_{23}	*1*
$[24, 2]$	C_{24}	*1*
$[24, 5]$	$C_4 \times S_3$	*1*
$[24, 6]$	D_{24}	-
$[24, 9]$	$C_{12} \times C_2$	*14*
$[24, 10]$	$C_3 \times D_8$	*7*
$[24, 11]$	$C_3 \times Q_8$	-
$[24, 13]$	$C_2 \times A_4$	*8*
$[24, 14]$	$C_2 \times C_2 \times S_3$	*10*
$[24, 15]$	$C_6 \times C_2 \times C_2$	*136*
$[25, 1]$	C_{25}	*1*
$[25, 2]$	$C_5 \times C_5$	*16*
$[26, 2]$	C_{26}	*1*
$[27, 1]$	C_{27}	*1*
$[27, 2]$	$C_9 \times C_3$	*20*
$[27, 3]$	$(C_3 \times C_3) \rtimes C_3$	*22*
$[27, 4]$	$C_9 \rtimes C_3$	*4*
$[27, 5]$	$C_3 \times C_3 \times C_3$	*202*

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IdGroup(G)	*StructureDescription*(G)	$m(G)$
$[28, 2]$	C_{28}	*1*
$[28, 3]$	D_{28}	*1*
$[28, 4]$	$C_{14} \times C_2$	*9*
$[29, 1]$	C_{29}	*1*
$[30, 4]$	C_{30}	*1*
$[31, 1]$	C_{31}	*1*

The following theorem follows from Lemma 1.

Theorem 1. *Let G be a group of order less than 32. If G is an additive group of nearring with identity R, then G is endocyclic.*

3. Local nearrings and endocyclic groups. Main results about local nearrings are given in surveys [9] and [10].

Definition 3. *A nearring with identity is called local if the set of all its non-invertible elements is a subgroup of its additive group.*

The LocalNR package provides functionality for investigating nearrings with identity and local nearrings. It also includes the *AllLocalNearRings* database of local nearrings of small orders. In this library, local nearrings are arranged according to their additive and multiplicative groups, for example, the command

$$N := AllLocalNearRings(k, l, m, n)$$

defines the lists of local nearrings N. The arguments k, l, m, n are from IdGroup of the additive group and the multiplicative group, respectively.

3.1. Local nearrings and endocyclic groups of order 32. Let $n(G)$ be the number of all non-isomorphic local nearrings R whose additive group R^+ is isomorphic to the group G.

There exist 51 non-isomorphic groups of order $32 = 2^5$ from which 35 are endocyclic groups and only 19 of these groups are the additive groups of local nearrings. The list and the database of all local nearrings of order 32 can be found in [11], the LocalNR package and [12].

Lemma 2. *The following groups G of order 32 are endocyclic. Moreover, if G is an additive group of a local nearring R, then the following holds.*

IdGroup(G)	*StructureDescription*(G)	$n(G)$
$[32, 1]$	C_{32}	*1*
$[32, 2]$	$(C_4 \times C_2) \rtimes C_4$	*1397*
$[32, 3]$	$C_8 \times C_4$	*880*
$[32, 4]$	$C_8 \rtimes C_4$	*798*
$[32, 5]$	$(C_8 \times C_2) \rtimes C_2$	*1945*
$[32, 6]$	$((C_4 \times C_2) \rtimes C_2) \rtimes C_2$	*433*
$[32, 7]$	$(C_8 \rtimes C_2) \rtimes C_2$	*225*
$[32, 8]$	$C_2.((C_4 \times C_2) \rtimes C_2) = (C_2 \times C_2).(C_4 \times C_2)$	*208*

Continued on the next page

IdGroup(G)	*StructureDescription*(G)	$n(G)$
$[32, 9]$	$(C_8 \times C_2) \rtimes C_2$	-
$[32, 10]$	$Q_8 \rtimes C_4$	-
$[32, 11]$	$(C_4 \times C_4) \rtimes C_2$	-
$[32, 12]$	$C_4 \rtimes C_8$	*2406*
$[32, 16]$	$C_{16} \times C_2$	*129*
$[32, 17]$	$C_{16} \rtimes C_2$	*129*
$[32, 18]$	D_{32}	-
$[32, 21]$	$C_4 \times C_4 \times C_2$	*135558*
$[32, 22]$	$C_2 \times ((C_4 \times C_2) \rtimes C_2)$	*149374*
$[32, 23]$	$C_2 \times (C_4 \rtimes C_4)$	*157905*
$[32, 24]$	$(C_4 \times C_4) \rtimes C_2$	*262544*
$[32, 25]$	$C_4 \times D_8$	-
$[32, 26]$	$C_4 \times Q_8$	-
$[32, 27]$	$(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2$	-
$[32, 28]$	$(C_4 \times C_2 \times C_2) \rtimes C_2$	-
$[32, 34]$	$(C_4 \times C_4) \rtimes C_2$	-
$[32, 35]$	$C_4 \rtimes Q_8$	-
$[32, 36]$	$C_8 \times C_2 \times C_2$	*177175*
$[32, 37]$	$C_2 \times (C_8 \rtimes C_2)$	*527419*
$[32, 39]$	$C_2 \times D_{16}$	-
$[32, 43]$	$(C_2 \times D_8) \rtimes C_2$	-
$[32, 45]$	$C_4 \times C_2 \times C_2 \times C_2$	*>6684533*
$[32, 46]$	$C_2 \times C_2 \times D_8$	-
$[32, 47]$	$C_2 \times C_2 \times Q_8$	-
$[32, 49]$	$(C_2 \times D_8) \rtimes C_2$	-
$[32, 50]$	$(C_2 \times Q_8) \rtimes C_2$	-
$[32, 51]$	$C_2 \times C_2 \times C_2 \times C_2 \times C_2$	*>7007053*

As a consequence of Lemma 2 we have the following result.

Theorem 2. *Let G be a group of order 32. If G is an additive group of a local nearring R, then G is endocyclic.*

3.2. Local nearrings and endocyclic 2-generated groups of order 64. There exist 267 non-isomorphic groups of order $64 = 2^6$ from which 53 are 2-generated groups and only 39 of these groups are endocyclic. Moreover, 24 of these groups are the additive groups of local nearrings. The list and the database of all local nearrings of order 64 can be found in [10], the LocalNR package and [13].

Lemma 3. *The following 2-generated groups G of order 64 are endocyclic. Moreover, if G is an additive group of a local nearring R, then the following holds.*

IdGroup(G)	*StructureDescription*(G)	$n(G)$
$[64, 2]$	$C_8 \times C_8$	*1683*
$[64, 3]$	$C_8 \rtimes C_8$	*1628*
$[64, 4]$	$((C_8 \times C_2) \rtimes C_2) \rtimes C_2$	*167245*

Continued on the next page

$IdGroup(G)$	$StructureDescription(G)$	$n(G)$
$[64, 5]$	$(C_4 \times C_2) \rtimes C_8$	*118190*
$[64, 6]$	$(C_8 \times C_4) \rtimes C_2$	-
$[64, 7]$	$Q_8 \rtimes C_8$	-
$[64, 8]$	$((C_8 \times C_2) \rtimes C_2) \rtimes C_2$	-
$[64, 9]$	$(C_2 \times Q_8) \rtimes C_4$	-
$[64, 10]$	$(C_8 \rtimes C_4) \rtimes C_2$	*7424*
$[64, 11]$	$(C_2 \times C_2).((C_4 \times C_2) \rtimes C_2) =$ $= (C_4 \times C_2).(C_4 \times C_2)$	-
$[64, 12]$	$(C_4 \rtimes C_8) \rtimes C_2$	*5633*
$[64, 14]$	$(C_2 \times C_2).((C_4 \times C_2) \rtimes C_2) =$ $= (C_4 \times C_2).(C_4 \times C_2)$	*5520*
$[64, 15]$	$C_8 \rtimes C_8$	*2384*
$[64, 16]$	$C_8 \rtimes C_8$	*2384*
$[64, 17]$	$(C_8 \times C_2) \rtimes C_4$	*433060*
$[64, 18]$	$(C_8 \times C_2) \rtimes C_4$	-
$[64, 19]$	$C_4.(C_4 \times C_4)$	-
$[64, 20]$	$(C_4 \times C_4) \rtimes C_4$	-
$[64, 21]$	$(C_8 \times C_2) \rtimes C_4$	-
$[64, 23]$	$(C_4 \times C_2 \times C_2) \rtimes C_4$	*111758*
$[64, 24]$	$(C_8 \rtimes C_2) \rtimes C_4$	*109189*
$[64, 26]$	$C_{16} \times C_4$	*11467*
$[64, 27]$	$C_{16} \rtimes C_4$	*11467*
$[64, 28]$	$C_{16} \rtimes C_4$	-
$[64, 29]$	$(C_{16} \times C_2) \rtimes C_2$	*28185*
$[64, 30]$	$(C_{16} \rtimes C_2) \rtimes C_2$	*4433*
$[64, 32]$	$((C_8 \rtimes C_2) \rtimes C_2) \rtimes C_2$	-
$[64, 33]$	$(C_4 \times C_2 \times C_2) \rtimes C_4$	-
$[64, 34]$	$(((C_4 \times C_2) \rtimes C_2) \rtimes C_2) \rtimes C_2$	*16177*
$[64, 35]$	$(C_4 \times C_4) \rtimes C_4$	*15504*
$[64, 36]$	$(C_2.((C_4 \times C_2) \rtimes C_2) =$ $(C_2 \times C_2).(C_4 \times C_2)) \rtimes C_2$	*15761*
$[64, 37]$	$C_2.(((C_4 \times C_2) \rtimes C_2) \rtimes C_2) =$ $(C_4 \times C_2).(C_4 \times C_2)$	*15920*
$[64, 38]$	$(C_{16} \times C_2) \rtimes C_2$	-
$[64, 41]$	$(C_{16} \rtimes C_2) \rtimes C_2$	-
$[64, 44]$	$C_4 \rtimes C_{16}$	*28500*
$[64, 45]$	$C_8.D_8 = C_4.(C_8 \times C_2)$	*1920*
$[64, 50]$	$C_{32} \times C_2$	*257*
$[64, 51]$	$C_{32} \rtimes C_2$	*257*
$[64, 52]$	D_{64}	-

The following result follows from Lemma 3.

Theorem 3. *Let G be a 2-generated group of order 64. If G is an additive group of a local nearring R, then G is endocyclic.*

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Раєвська І., Раєвська М. Локальні майже-кільця та ендоциклічні групи малих порядків.

Майже-кільця — це узагальнення асоціативних кілець, в яких адитивна група може бути неабелевою, а додавання пов'язане з множенням хоча б одним дистрибутивним законом, лівим або правим. У цьому сенсі локальні майже-кільця є узагальненням локальних кілець.

Пакет SONATA системи комп'ютерної алгебри GAP містить бібліотеку всіх неізоморфних майже-кілець порядку не більше 15 та майже-кілець з одиницею порядку не більше 31, серед яких 698 є локальними. Однак класифікація майже-кілець вищих порядків вимагає набагато складніших обчислень.

Наведено класифікацію `майже-кілець з одиницею` (до порядку 32) та `локальних майже-кілець` (порядків 32 та 64) на `ендоциклічних адитивних групах`. Крім того, представлено деяку інформацію щодо цих майже-кілець з `GAP-пакетів LocalNR` та `SONATA`.

Ключові слова: майже-кільце з одиницею, локальне майже-кільце, ендоциклічна група, адитивна група, пакет LocalNR, GAP.

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