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ON CERTAIN DISCRETE QUANTUM DISTRIBUTIONS

The article investigates the discrete q-distributions – the q-Binomial, q-Negative Binomial, and q-Poisson – which play an important role in quantum calculus, q-combinatorics, and the theory of special functions. Based on the application of q-derivatives and the q-Taylor formula, a unified approach is proposed for constructing recurrence relations for the initial and central moments of these distributions. Explicit formulas for low-order moments are obtained, generalizing classical results and correctly reducing to them when $q = 1$. The proposed method allows for the systematization of known fragmentary results and provides a basis for further research into q-probabilistic models, particularly in connection with q-orthogonal polynomials and stochastic processes.

Keywords: discrete quantum distributions, q-Binomial distribution, q-Negative Binomial distribution, q-Poisson distribution, q-calculus, moments of distributions.

1. Introduction. Quantum calculus (or q-calculus) is one of the important directions in modern mathematics, which emerged as a generalization of classical differential calculus and has found wide application in combinatorics, the theory of special functions, probability theory, and mathematical physics. q-analogs of discrete distributions attract significant attention from researchers because they are naturally connected with q-combinatorics and allow for the modeling of random processes with non-classical symmetries or dependencies that arise in quantum and stochastic systems.

The construction of q-distributions and the study of their properties have been actively developed over the last decades. Discrete probability distributions play a key role in modeling stochastic processes and analyzing random variables. The classical properties of such distributions, particularly the Binomial, Negative Binomial, and Poisson distributions, are systematically detailed in the monograph by Johnson, Kemp, and Kotz [1]. However, contemporary research increasingly turns to q-generalizations of these models, which naturally arise in the context of quantum calculus, q-combinatorics, and the theory of special functions.

The basic concepts of q-calculus — q-numbers, q-factorials, q-binomial coefficients, q-derivatives — are thoroughly discussed in the work by Kac and Cheung [4].

These tools form the basis for constructing q-analogs of classical distributions. The first systematic studies of the q-Binomial and q-Poisson distributions were presented in the work by Kupershmidt [3]. A much wider range of q-distributions is described in the monograph by Charalambides [2], where their role in statistics and combinatorics is emphasized.

The further development of the theory of q-distributions is related to the study of basic hypergeometric series [5] and q-orthogonal polynomials [6], in which moments play a fundamental role. Separate stochastic interpretations and generalizations of q-models are given in the works by Floreanini and Vinet [7]. Despite a significant number of results, a consistent approach to systematically obtaining the moments of the main q-distributions is still lacking in the literature, and the available formulas are often presented fragmentarily.

The goal of this work is to derive recurrence relations for the moments of the q-Binomial, q-Negative Binomial, and q-Poisson distributions based on q-derivatives and the q-Taylor formula, and to obtain explicit expressions for the initial and central moments of low orders.

2. Main result. In this section, the main results of the paper are presented, together with the necessary statements and lemmas used for constructing q-distributions and deriving recurrence relations for their moments. The presented relations ensure the correctness of the corresponding probabilistic models and allow a systematic application of q-derivatives and the q-Taylor formula.

We will use the notation and basic facts of quantum calculus (q-calculus) from the book [3]:

$$[n] = [n]_q := \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}, \quad q > 0, \quad [0] := 0,$$

$$[n]! = [n][n-1]\ldots[3][2][1], \quad [0]! := 1, \quad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!} = \begin{bmatrix} n \\ n-k \end{bmatrix},$$

$$[a+b]_q^n = (a+b)(a+qb)(a+q^2b)\ldots(a+q^{n-1}b),$$

$$\mathrm{D}_q f(x) := \frac{f(x) - f(qx)}{x(1-q)}, \quad \mathrm{e}_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]!},$$

$$\mathrm{e}_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]!}, \quad \mathrm{D}_q(\mathrm{e}_q(ax)) = a\,\mathrm{e}_q(ax), \quad \mathrm{D}_{1/q}(\mathrm{e}_q(x)) = \mathrm{e}_q(q^{-1}x),$$

$$\text{the } q\text{-Taylor formula is:} \quad f(x) = \sum_{k=0}^{\infty} \frac{(x-c)_q^k}{[k]!} \mathrm{D}_q^k f(x)\Big|_{x=c}.$$

We begin with a normalizing identity for the q-binomial distribution, which guarantees that the corresponding probability mass function defines a valid probability distribution.

Lemma 1.

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k (1-x)_q^{n-k} = 1. \tag{1}$$

Proof. We apply the q-Taylor formula to the function $f(x) = x^n$, taking $c = 1$. We have

$$x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (x-1)_q^k.$$

Replacing x with $1/x$, we obtain

$$\frac{1}{x^n} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \left(\frac{1}{x} - 1\right)\left(\frac{1}{x} - q\right)\dots\left(\frac{1}{x} - q^{k-1}\right) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^{-k}(1-x)_q^k,$$

whence

$$1 = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k}(1-x)_q^k,$$

which is equivalent to (1).

The following lemma establishes an analogue of the normalization condition for the negative q-binomial distribution and plays a key role in the construction of the corresponding probabilistic model.

Lemma 2.

$$\sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} q^{k(k-1)/2} \frac{x^k}{(1+x)_q^{n+k}} = 1, \ x > 0, \ 0 < q < 1, \ n > 0. \qquad (2)$$

Proof. We apply the q-Taylor formula to the function $f(t) = \frac{1}{(1-t)_q^n}$. We have

$$\frac{1}{(1-t)_q^n} = \sum_{k=0}^{\infty} \frac{(t-c)_q^k}{[k]!} \mathrm{D}_q^k \frac{1}{(1-t)_q^n}\Big|_{t=c} = \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{(t-c)_q^k}{(1-c)_q^{n+k}}.$$

Since $(t-c)_q^k = (t-c)(t-qc)\dots(t-q^{k-1}c)$, for $t = 0$ and $c = -x$,

$$(t-c)_q^k = q^{k(k-1)/2}x^k, \quad \text{and thus} \quad \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} q^{k(k-1)/2} \frac{x^k}{(1+x)_q^{n+k}}.$$

The next lemma presents an identity related to the q-exponential function, which underlies the construction of the q-Poisson distribution.

Lemma 3.

$$\sum_{k=0}^{\infty} q^{-k(k-1)/2} \frac{x^k}{[k]!} \mathrm{e}_q(-xq^{-k}) = 1 \qquad (3)$$

Proof. We apply the q-Taylor formula to the function $f(t) = \mathrm{e}_q(-t)$. We have

$$\mathrm{e}_q(-t) = \sum_{k=0}^{\infty} \frac{(t-x)_{1/q}^k}{[k]_{1/q}!} \mathrm{D}_{1/q}^k \mathrm{e}_q(-t)\Big|_{t=x} = \sum_{k=0}^{\infty} \frac{(t-x)_{1/q}^k}{[k]_{1/q}!} (-1)^k q^{-k(k-1)/2} \mathrm{e}_q(-q^{-k}t)\Big|_{t=x}.$$

Taking $t = 0$ in the last sum, and noting that $(-x)_{1/q}^k = (-1)^k q^{k(k-1)/2} x^k$ and $[k]_{1/q}! = q^{-k(k-1)/2}[k]!$, we obtain (3).

Lemmas 1–3 establish the main normalization relations for the corresponding q-distributions and guarantee the non-negativity and unit-sum property of the probability mass functions.

The expressions under the summation signs in formulas (1), (2), (3) are non-negative, and therefore they can be used for constructing probability distributions of random variables.

Definition 1. *A random variable ξ is said to have a q-Binomial distribution with parameters n and x if*

$$\Pr\{\xi = [k]\} = \begin{bmatrix} n \\ k \end{bmatrix} x^k (1-x)_q^{n-k},\ 0 < x < 1,\ 0 < q < 1,\ k = 0, 1, 2, \ldots, n.$$

We will denote this fact as: $\xi \subset B_q(n, x)$.

Definition 2. *A random variable ξ is said to have a q-Negative Binomial distribution with parameters n and x if*

$$\Pr\left\{\xi = \frac{[k]}{q^{k-1}}\right\} = \begin{bmatrix} n+k-1 \\ k \end{bmatrix} q^{k(k-1)/2} \frac{x^k}{(1+x)_q^{n+k}},\ x > 0,\ 0 < q < 1,\ k = 0, 1, 2, \ldots$$

We will denote this fact as: $\xi \subset NB_q(n, x)$.

Definition 3. *A random variable ξ is said to have a q-Poisson distribution with parameter x if*

$$\Pr\{\xi = [k]\} = q^{-k(k-1)/2} \frac{x^k}{[k]!} \mathrm{e}_q(-xq^{-k}),\ x > 0,\ 0 < q < 1,\ k = 0, 1, 2, \ldots$$

We will denote this fact as: $\xi \subset P_q(x)$.

The following lemmas are used to derive recurrence relations for the moments of the corresponding q-distributions based on the properties of q-derivatives.

Lemma 4.

$$x(1-x)\mathrm{D}_q(x^k(1-x)_q^{n-k}) = x^k(1-x)_q^{n-k}([k] - [n]x). \tag{4}$$

Proof. $\mathrm{D}_q(x^k(1-x)_q^{n-k}) =$

$$\frac{x^k(1-x)(1-qx)\ldots(1-q^{n-k-1}x) - q^k x^k(1-qx)(1-q^2x)\ldots(1-q^{n-k}x)}{x(1-q)} =$$

$$= x^{k-1}(1-qx)\ldots(1-q^{n-k-1}x)([k] - [n]x),$$

and from this, (4) follows.

Lemma 5.

$$\frac{x(x+q)}{q}\mathrm{D}_{1/q}\frac{x^k}{(1+x)_q^{n+k}} = \frac{x^k}{(1+x)_q^{n+k}}\left(\frac{[k]}{q^{k-1}} - [n]x\right) \tag{5}$$

Proof.

$$\mathrm{D}_{1/q}\frac{x^k}{(1+x)_q^{n+k}} = \frac{1}{x(1-1/q)} =$$

$$= \left(\frac{x^k}{(1+x)(1+qx)\dots(1+q^{n+k-1}x)} - \frac{x^k}{(1+x/q)(1+x)\dots(1+q^{n+k-2}x)} \right) =$$

$$\frac{q}{x(q-1)} \frac{x^k}{(1+x)(1+qx)\dots(1+q^{n+k-2}x)} \left(\frac{1}{1+q^{n+k-1}x} - \frac{1}{q^k(1+x/q)} \right) =$$

$$\frac{q}{x(q-1)} \frac{x^k}{(1+x)(1+qx)\dots(1+q^{n+k-2}x)} \left(\frac{q^k + xq^{k-1} - 1 - q^{n+k-1}x}{(1+q^{n+k-1}x)(q+x)q^{k-1}} \right) =$$

$$\frac{q}{x(q+x)} \frac{x^k}{(1+x)_q^{n+k}} \left(\frac{(q^k-1)/(q-1) - xq^{k-1}(1-q^n)/(1-q)}{(1+q^{n+k-1}x)q^{k-1}} \right) =$$

$$\frac{q}{x(q+x)} \frac{x^k}{(1+x)_q^{n+k}} \left(\frac{[k]q^{1-k} - [n]x}{(1+q^{n+k-1}x)(q+x)} \right),$$

and from this, (5) follows.

Lemma 6.

$$x\mathrm{D}_q(x^k\mathrm{e}_q(-q^{-k}x)) = x^k\mathrm{e}_q(-q^{-k}x)([k]-x) \tag{6}$$

Proof. By the rules for finding the q-derivative of a product of two functions, we have:

$$\mathrm{D}_q(x^k\mathrm{e}_q(-q^{-k}x)) = q^k x^k \mathrm{D}_q\mathrm{e}_q(-q^{-k}x) + \mathrm{e}_q(-q^{-k}x)\mathrm{D}_q x^k,$$

and from this, (6) follows.

Let $m \in \mathbb{N}$, and $a \in \mathbb{R}$. We denote by $s_m(a)$ the expectation of the random variable ξ, that is, $s_m(a) = \mathrm{E}(\xi - a)^m$. Then $s_m(0) = \alpha_m$ are the initial moments of order m, and $s_m(\alpha_1) = \mu_m$ are the central moments of order m.

Using the lemmas presented above, we proceed to establish recurrence formulas for the raw and central moments of the main q-distributions.

Theorem 1. *If $\xi \subset B_q(n,x)$, then*

$$s_{m+1}(a) = x(1-x)\mathrm{D}_q s_m(a) + ([n]x - a)s_m(a), \;\; s_0(a) = 1, \;\; s_1(a) = [n]x - a. \tag{7}$$

If $\xi \subset NB_q(n,x)$, then

$$s_{m+1}(a) = x(x/q+1)\mathrm{D}_{1/q} s_m(a) + ([n]x - a)s_m(a), \;\; s_0(a) = 1, \;\; s_1(a) = [n]x - a. \tag{8}$$

If $\xi \subset P_q(x)$, then

$$s_{m+1}(a) = x\mathrm{D}_q s_m(a) + (x-a)s_m(a), \;\; s_0(a) = 1, \;\; s_1(a) = x - a. \tag{9}$$

Proof. Let the function $f(x)$ be defined on the set of values of the random variable ξ. Then for $\xi \subset B_q(n,x)$, by virtue of Lemma 4,

$$x(1-x)\mathrm{D}_q\mathrm{E}f(\xi) = \sum_{k=0}^{n} f([k]) \begin{bmatrix} n \\ k \end{bmatrix} \mathrm{D}_q(x^k(1-x)_q^{n-k}) =$$

$$\sum_{k=0}^{n} f([k]) \begin{bmatrix} n \\ k \end{bmatrix} x^k(1-x)_q^{n-k}([k] - a + a - [n]x) = \mathrm{E}((\xi - a)f(\xi)) + (a - [n]x)\mathrm{E}f(\xi).$$

Hence

$$\mathrm{E}((\xi - a)f(\xi)) = x(1-x)\mathrm{D}_q\mathrm{E}f(\xi) + ([n]x - a)\mathrm{E}f(\xi).$$

In this relation, we substitute $(x-a)^m$ for $f(x)$ to obtain (7).

If $\xi \subset NB_q(n,x)$, then by virtue of Lemma 5,

$$\frac{x(x+q)}{q}\mathrm{D}_{1/q}\mathrm{E}f(\xi) = \sum_{k=0}^{\infty} f\left(\frac{[k]}{q^{k-1}}\right)\begin{bmatrix} n+k-1 \\ k \end{bmatrix}\frac{x(x+q)}{q}\mathrm{D}_{1/q}\frac{x^k}{(1+x)_q^{n+k}} =$$

$$\sum_{k=0}^{\infty} f\left(\frac{[k]}{q^{k-1}}\right)\begin{bmatrix} n+k-1 \\ k \end{bmatrix}\frac{x^k}{(1+x)_q^{n+k}}\left(\frac{[k]}{q^{k-1}} - a + a - [n]x\right) =$$

$$= \mathrm{E}((\xi-a)f(\xi)) + (a-[n]x)\mathrm{E}f(\xi).$$

Hence

$$\mathrm{E}((\xi-a)f(\xi)) = \frac{x(x+q)}{q}\mathrm{D}_{1/q}\mathrm{E}f(\xi) + ([n]x-a)\mathrm{E}f(\xi).$$

In this relation, we substitute $(x-a)^m$ for $f(x)$ to obtain (8).

If $\xi \subset P_q(x)$, then by virtue of Lemma 6,

$$x\mathrm{D}_q\mathrm{E}f(\xi) = \sum_{k=0}^{\infty} f([k])q^{-k(k-1)/2}\frac{x}{[k]!}\mathrm{D}_q(x^k \mathrm{e}_q(-q^{-k}x)) =$$

$$\sum_{k=0}^{\infty} f([k])q^{-k(k-1)/2}\frac{1}{[k]!}\mathrm{D}_q(x^k \mathrm{e}_q(-q^{-k}x)) =$$

$$\sum_{k=0}^{\infty} f([k])q^{-k(k-1)/2}\frac{x^k}{[k]!}\mathrm{e}_q(-q^{-k}x)([k]-a+a-x) = \mathrm{E}((\xi-a)f(\xi)) + (a-x)\mathrm{E}f(\xi).$$

Hence

$$\mathrm{E}((\xi-a)f(\xi)) = x\mathrm{D}_q\mathrm{E}f(\xi) + (x-a)\mathrm{E}f(\xi).$$

In this relation, we substitute $(x-a)^m$ for $f(x)$ to obtain (9).

Corollary 1. *If $\xi \subset B_q(n,x)$, then*

$$s_2(a) = a^2 + [n](1-2a)x + [n]([n]-1)x^2,$$

$$s_3(a) = -a^3 + (1-3a+3a^2)[n]x + ([n]-1)[n](1+[2]-3a)x^2 + ([n]-1)[n]([n]-[2])x^3,$$

$$s_4(a) = x(1-x)(1-3a+3a^2)[n] + ([n]-1)[n]((1+[2]-3a)[2]x+$$
$$([n]-1)[n]([n]-[2]))[3]x^2 + ([n]x-a)s_3(a),$$

In particular,

$$\alpha_2 = [n]x + [n]([n]-1)x^2, \quad \mu_2 = [n]x(1-x),$$

$$\alpha_3 = [n]x + ([n]-1)[n](1+[2])x^2 + ([n]-1)[n]([n]-[2])x^3,$$

$$\mu_3 = [n]x - [n](1+[2][n]+[2]-[2][n])x^2 + [n]([2][n]-2[n]+[2])x^3,$$

$$\alpha_4 = [n]x + ([n]-1)[n](1+[2]+[2]^2)x^2 + ([n]-1)[n]([n]-[2])(1+[2]+[3])x^3+$$
$$+([n]-1)[n]([n]-[2])([n]-[3])x^4,$$

$$\mu_4 = x(1-x)([n]-1)[n]([n]-[2])[3]x^2 + ([n]-1)[n](1+[2]-3[n]x)+$$

$$+[n](1-3[n]x+3[n]^2x^2).$$

If we set $q=1$ in these formulas, we obtain the known formulas for the moments of the ordinary Binomial distribution [1, p. 110].

$$\alpha_2 = nx(1-x+nx), \quad \mu_2 = nx(1-x),$$

$$\alpha_3 = nx(1-3x+3nx+(n-1)(n-2)x^2), \quad \mu_3 = nx(1-x)(1-2x),$$

$$\alpha_4 = nx(1+7(n-1)x+6(n-1)(n-2)x^2+(n-1)(n-2)(n-3)x^3),$$

$$\mu_4 = nx(1-x)(1-6x+6x^2+3nx-3nx^2).$$

Corollary 2. *If $\xi \subset NB_q(n,x)$, then*

$$s_2(a) = a^2+[n](1+2a)x+[n]([n]+1/q)x^2,$$

$$s_3(a) = -a^3+(1-3a+3a^2)[n]x+([n]q+1)[n](q+[2]+3aq)q^{-2}x^2+$$

$$+([n]q+1)[n]([n]q^2+[2])q^{-3}x^3,$$

$$s_4(a) = x(1+x/q)(1-3a+3a^2)[n]+([n]q+1)[n]((q+[2]+3aq)[2]q^{-2}x+$$

$$+([n]q+1)[n]([n]q^2+[2]))[3]q^{-5}x^2+([n]x-a)s_3(a),$$

In particular,

$$\alpha_2 = [n]x+[n]([n]+1/q)x^2, \ \mu_2 = [n]x(1+x/q),$$

$$\alpha_3 = [n]x+([n]q+1)[n](q+[2])q^{-2}x^2+([n]q+1)[n]([n]q^2+[2])q^{-3}x^3,$$

$$\mu_3 = [n]x-[n](-q+2[n]q^2+[2]-[2][n]q)q^{-2}x^2+[n](2[n]q^2-[2]-[2][n]q)q^{-3}x^3,$$

$$\alpha_4 = [n]x+([n]q+1)[n](q^2+[2]q+[2]^2)q^{-3}x^2+([n]q+1)[n]([n]q^2+$$

$$[2])(q^2+[2]q+[3])q^5x^3+([n]q+1)[n]([n]q^2+[2])([n]q^3+[3])q^{-6}x^4,$$

$$\mu_4 = [n]x+\left(-4[n]^2+[n](1/q+[n])+[n](1/q+[n])[2]q^{-1}+[n](1/q+[n])[2]^2q^{-2}\right)x^2$$

$$+(6[n]^3-4[n]^2(1+[n]q)q^{-1}-4[n]^2(1+[n]q)[2]q^{-2}+[n](1+[n]q)([n]q^2+[2])q^{-3}+$$

$$+[n](1+[n]q)([n]q^2+[2])[2]q^{-4}+[n](1+[n]q)([n]q^2+[2])[3]q^{-5})x^3+$$

$$+[n](1+[n]q)([n]q^2+[2])([n]q^3+[3])q^{-6}x^4$$

If we set $q=1$ in these formulas, we obtain the known formulas for the moments of the ordinary Negative Binomial distribution [1, p. 316].

$$\alpha_2 = nx(1+x+nx), \quad \mu_2 = nx(1+x),$$

$$\alpha_3 = nx(1+3(n+1)x+(n+1)(n+2)x^2), \quad \mu_3 = nx(1+x)(1+2x),$$

$$\alpha_4 = nx(1+7(n+1)x+6(n+1)(n+2)x^2+(n+1)(n+2)(n+3)x^3),$$

$$\mu_4 = 3n^2x^2(1+x)^2+nx(1+x)(1+6x+6x^2).$$

Corollary 3. *If $\xi \subset P_q(x)$, then*

$$s_2(a) = x + (x-a)^2, \quad s_3(a) = -a^3 + (1-3a-3a^2)x + (1+[2]-3a)x^2 + x^3,$$

$$s_4(a) = a^4 + (-a+3a^2-4a^3+(1-3a+3a^2))x + (1-4a+6a^2-[2]a+ \\ 1+[2]-3a)[2]x^2 + (1-4a+[2]+[3])x^3 + x^4,$$

$$s_5(x) = -a^5 + (1-5a+10a^2-10a^3+5a^4)x + (1-5a+10a^2-10a^3+[2]-5[2]a+ \\ 10[2]a^2+[2]^2-5[2]^2a+[2]^3)x^3 + (1-5a+10a^2+[2]-5[2]a+[2]^2+[3] \\ -5[3]a+[2][3]+[3]^3)x^3 + (1+[2]+[3]+[4])x^4 + x^5.$$

In particular,

$$\alpha_2 = x + x^2, \quad \mu_2 = x,$$

$$\alpha_3 = x + (1+[2])x^2 + x^3, \quad \mu_3 = x(1-2x+[2])x,$$

$$s_2(a) = x + (x-a)^2,$$

$$s_3(a) = -a^3 + (1-3a-3a^2)x + (1+[2]-3a)x^2 + x^3,$$

$$s_4(a) = a^4 + (-a+3a^2-4a^3+(1-3a+3a^2))x + (1-4a+6a^2-[2]a+ \\ 1+[2]-3a)[2]x^2 + (1-4a+[2]+[3])x^3 + x^4,$$

$$s_5(x) = -a^5 + (1-5a+10a^2-10a^3+5a^4)x + (1-5a+10a^2-10a^3+[2]-5[2]a+ \\ 10[2]a^2+[2]^2-5[2]^2a+[2][3])x^3 + (1-5a+10a^2+[2]-5[2]a+[2]^2+[3] \\ -5[3]a+[2][3])x^3 + (1+[2]+[3]+[4])x^4 + x^5.$$

$$\alpha_4 = x + (1+(1+[2])[2])x^2 + (1+[2]+[3])x^3 + x^4,$$

$$\mu_4 = x + (-3+[2]+[2]^2)x^2 + (3-3[2]+[3])x^3,$$

$$\alpha_5 = x + (1+[2](1+[2]+[2]^2))x^2 + (1+[2]+[2]^2+(1+[2]+[3](1+[2]+[3]))x^3+ \\ (1+[2]+[3]+[4])x^4 + x^5,$$

$$\mu_5 = x + (-4+[2]+[2]^2+[2]^3)x^2 + (6-4[2]-4[2]^2+[3]+[2][3]+[3]^2)x^3+ \\ (-4+6[2]-4[3]+[4])x^4.$$

If we set $q = 1$ in these formulas, we obtain the known formulas for the moments of the ordinary Poisson distribution [1, p. 163]:

$$\alpha_2 = x + x^2, \quad \mu_2 = x,$$

$$\alpha_3 = x + 3x^2 + x^3, \quad \mu_3 = x,$$

$$\alpha_4 = x + 7x^2 + 6x^3 + x^4, \quad \mu_4 = x + 3x^2,$$

$$\alpha_5 = x + 15x^2 + 25x^3 + 10x^4 + x^5, \quad \mu_5 = x + 10x^2.$$

3. Conclusions and prospects for further research. The article proposes a unified approach to constructing the moments of the main discrete q-distributions — the q-Binomial, q-Negative Binomial, and q-Poisson — based on the framework of q-derivatives and the q-Taylor formula. The derived recurrence relations for the initial and central moments provide a generalization of the classical results and are logically recovered in the limit as $q = 1$. The presented explicit formulas for low-order moments demonstrate the effectiveness of the proposed method and allow for the systematization of results found fragmentarily in the literature.

The obtained results are useful for the further development of the theory of q-distributions, particularly in the context of stochastic models related to quantum calculus, q-combinatorics, and the theory of special functions. Prospects for future research include deriving asymptotic estimates of moments, analyzing their behavior under different regimes of the parameter q, applying the resulting formulas to problems of stochastic process modeling and the theory of random permutations, as well as employing q-distributions for constructing models of noisy data and new regularization methods in machine learning that ensure robustness to sample noise and anomalous observations.

Conflict of Interest

The authors declare that they have no conflicts of interest in relation to the current study, including financial, personal, authorship, or any other, that could affect the study, as well as the results reported in this paper.

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Data Availability

All data are available, either in numerical or graphical form, in the main text of the manuscript.

Use of artificial intelligence

The authors confirm that they did not use artificial intelligence technologies when creating the current work.

Contributions of authors

Volkov O. Yu.: Conceptualization, Formal analysis, Methodology, Writing — original draft. Volkov Yu. I.: Data curation, Supervision, Writing — review & editing. Voinalovych N. M.: Writing — review & editing.

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Волков О. Ю., Волков Ю. І., Войналович Н. М. Про деякі дискретні квантові розподіли.

У статті досліджуються дискретні q-розподіли — q-біноміальний, q-від'ємний біноміальний та q-пуассонівський — які відіграють важливу роль у квантовому численні, q-комбінаториці та теорії спеціальних функцій. На основі застосування q-похідних та q-формули Тейлора запропоновано єдиний підхід до побудови рекурентних співвідношень для початкових і центральних моментів зазначених розподілів. Отримано явні формули моментів низьких порядків, що узагальнюють класичні результати та коректно переходять до них при $q = 1$. Запропонований метод дозволяє систематизувати відомі фрагментарні результати та створює основу для подальших досліджень q-ймовірнісних моделей, зокрема у зв'язку з q-ортогональними многочленами та стохастичними процесами.

Ключові слова: дискретні квантові розподіли, q-біноміальний розподіл, від'ємний q-біноміальний розподіл; q-пуассонівський розподіл, q-числення, моменти розподілів.

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