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MATHEMATICAL MODELING OF THE SPATIO-TEMPORAL DISTRIBUTION OF SOLAR ENERGY POTENTIAL IN URBANIZED AREAS

The paper develops a mathematical model of the spatio-temporal distribution of solar energy potential in urbanized areas. Solar radiation is represented as a spatio-temporal function that combines an analytical description of solar irradiance with a corrective component accounting for spatial and temporal heterogeneity of the environment. The energy potential is formulated as an integral characteristic over the area of the territory and a given time interval, taking into account the temperature dependence of photovoltaic conversion efficiency. A numerical implementation of the model is proposed, and computational experiments are carried out using synthetic input data. It is shown that spatial heterogeneity of solar radiation can reduce the integrated energy potential by 14–29% compared to the homogeneous case, while accounting for the temperature factor leads to an additional reduction of 5–6%. The obtained results confirm the relevance of a spatio-temporal approach to the assessment of solar energy potential in urbanized areas.

Keywords: mathematical modeling, solar energy, spatio-temporal distribution, solar radiation, urbanized areas.

1. Introduction. Solar energy is a key component in the transition toward sustainable energy systems, particularly in urbanized areas where energy demand is concentrated and solar radiation exhibits complex spatial and temporal variability. However, the assessment of solar energy potential in such environments remains challenging due to the heterogeneity of building structures, surface properties, and meteorological conditions.

Conventional approaches to solar irradiance assessment are often based on averaged characteristics or treat radiation as a purely temporal function, which limits their applicability for detailed analysis and forecasting in urban contexts. In contrast, representing solar radiation as a spatio-temporal function of spatial coordinates and time enables the use of multivariable analysis, integral methods, and numerical modeling to obtain both local and aggregated estimates of energy potential.

Despite significant progress in applied solar energy research, a unified mathematical framework that integrates analytical representations of solar irradiance with spatio-temporal data remains insufficiently developed. This gap restricts systematic analysis and comparison of solar energy resources across different territories.

This work presents a mathematical model for the spatio-temporal distribution of solar energy in urban areas, using methods from mathematical analysis, multivariable functions, integral calculus, and numerical modeling. The approach treats solar energy potential as a spatio-temporal variable, offering a more informative framework than traditional models. The main innovation is a unified mathematical representation for assessing and optimizing photovoltaic systems in urban environments.

2. Analysis of Recent Studies and Publications. The assessment of solar energy potential has been widely studied in applied mathematics, energy engineering, and atmospheric physics. Classical approaches describe solar radiation using analytical models based on solar geometry, geographic location, and seasonal variability [1, 2], enabling its representation as a continuous function of time and location.

In recent years, research has increasingly relied on reanalysis and satellite-based data, which provide spatio-temporal discretization of solar radiation and allow it to be treated as a spatio-temporal field [3, 4]. This is particularly important for analyzing large and heterogeneous territories.

Urban environments represent a specific challenge due to pronounced spatial heterogeneity of solar irradiance caused by building geometry and surface properties. Studies [5, 6] show that this limits the applicability of classical averaged models and requires spatially explicit approaches.

Another important aspect is the dependence of photovoltaic conversion efficiency on temperature. This relationship has been studied in [7–9], where semi-empirical models linking efficiency to operating temperature have been proposed.

Despite extensive research, there is still no unified mathematical framework that integrates analytical models of solar irradiance with spatio-temporal data and enables the representation of solar energy potential as an integral spatio-temporal quantity. This gap motivates the development of hybrid models combining analytical formulations with observational data.

3. Problem Statement and Mathematical Model. Let us consider an urbanized territory, which in the mathematical sense is represented as a bounded domain $\Omega \subset \mathbb{R}^2$ corresponding to the horizontal projection of the area under study. The time interval of interest is denoted by $t \in [t_0, t_1]$, where t_0 and t_1 correspond to the beginning and the end of the observation period. Solar radiation incident on a unit surface area is treated as a spatio-temporal function $S = S(x, y, t)$, where $(x, y) \in \Omega$ are spatial coordinates and t denotes time. The function $S(x, y, t)$ characterizes the instantaneous density of the solar energy flux and may be specified analytically or reconstructed from spatio-temporal observational data.

Within the hybrid approach, solar radiation is represented as the product of analytical and corrective components:

$$S(x, y, t) = S_0(t, \varphi)\Phi(x, y, t), \quad (1)$$

where $S_0(t, \varphi)$ is the theoretical (astronomical) component of solar irradiance depending on time t and geographical latitude φ, and $\Phi(x, y, t)$ is a dimensionless corrective function accounting for atmospheric conditions, spatial heterogeneity, and specific features of the urban environment. The function $\Phi(x, y, t)$ satisfies the constraint $0 \leq \Phi(x, y, t) \leq 1$ and may be reconstructed using satellite or reanalysis data without an explicit analytical specification. In the latter case, $\Phi(x, y, t)$ can be defined as a normalized correction factor, for example, $\Phi(x, y, t) = \frac{S_{\text{obs}}(x,y,t)}{S_0(t,\varphi)}$, where $S_{\text{obs}}(x, y, t)$ denotes an observed or modeled solar radiation field. Such normalization ensures the dimensionless nature of Φ and preserves the physical interpretation of S_0 as an upper bound corresponding to ideal clear-sky conditions. Such a formulation allows solar radiation to be treated as a formalized spatio-temporal field without being tied to a specific data source.

The solar energy potential of the studied territory Ω at time t is defined as the integral of solar radiation weighted by the energy conversion efficiency:

$$E(t) = \int_{\Omega} S(x, y, t)\eta(x, y, t)dxdy. \tag{2}$$

Here $\eta(x, y, t)$ denotes the local photovoltaic conversion efficiency, which may depend on temperature and operating conditions.

To analyze the cumulative energy potential over the time interval $[t_0, t_1]$, the following integral characteristic is introduced:

$$E_{\text{tot}} = \int_{t_0}^{t_1} E(t)dt. \tag{3}$$

The efficiency of photovoltaic converters strongly depends on the operating surface temperature. Within the model, a linear approximation of the temperature dependence of efficiency is adopted:

$$\eta(T) = \eta_{\text{ref}}\left[1 - \beta\left(T - T_{\text{ref}}\right)\right], \tag{4}$$

where η_{ref} is the efficiency at the reference temperature T_{ref}, β is the temperature coefficient, and T is the temperature of the photovoltaic module surface.

In numerical experiments, temperature is defined parametrically as, $T = f(S(x, y, t))$, allowing thermal effects on conversion efficiency to be incorporated without an explicit heat transfer model. Alternatively, temperature may be treated as an external input from observational or modeled data without altering the formulation. The linear temperature dependence (4) is assumed valid within typical operating ranges of photovoltaic modules; outside this range, it should be considered an approximation requiring more complex nonlinear models.

The study aims to construct and analyze a spatio-temporal integral model of solar energy potential defined by (2)–(4) and to investigate its behavior under variations of spatial and temporal parameters. The model provides a basis for further analytical and numerical studies and enables comparative analysis across territories and time periods.

4. Analysis of the Properties of the Mathematical Model. Let us investigate the main properties of the proposed spatio-temporal model of solar energy potential, in particular its integral characteristics, temporal variability, and behavior under parameter variations.

We consider the mean value of the solar energy potential per unit area of the territory Ω at time t, defined as

$$\hat{E}(t) = \frac{1}{|\,\Omega\,|}\int_{\Omega} S(x, y, t)\eta(x, y, t)dxdy, \tag{5}$$

where $\mid \Omega \mid$ denotes the area of the domain Ω.

The quantity $\hat{E}(t)$ characterizes the averaged energy potential of the territory and allows comparison of different urbanized areas regardless of their size. Considering the representation of solar radiation in the form (1), we obtain

$$\hat{E}(t) = \frac{S_0(t,\varphi)}{\mid \Omega \mid} \int\limits_{\Omega} \Phi(x,y,t)\eta(x,y,t)dxdy. \tag{6}$$

Thus, the spatio-temporal variability of the averaged energy potential is determined by the product of the theoretical component of solar irradiance and an integral factor that accumulates the effects of spatial heterogeneity and conversion efficiency.

Considering the bounds $0 \le \Phi(x,y,t) \le 1$ and $0 \le \eta(x,y,t) \le \eta_{\max}$, where $\eta_{\max}$ is the maximum conversion efficiency, the following estimate holds for the averaged energy potential:

$$0 \le \hat{E}(t) \le S_0(t,\varphi)\eta_{\max}. \tag{7}$$

The limiting case $\Phi(x,y,t) \equiv 1$ and $\eta(x,y,t) \equiv \eta_{\max}$ corresponds to an idealized homogeneous territory without atmospheric or spatial losses, for which the model reduces to the classical analytical estimate of solar energy potential. Hence, the proposed model can be regarded as a generalization of well-known simplified approaches.

The dependence of $\hat{E}(t)$ on time is determined, on the one hand, by the seasonal periodicity of the theoretical irradiance $S_0(t,\varphi)$, and on the other hand, by the temporal variability of the corrective function $\Phi(x,y,t)$. Assuming sufficient temporal smoothness of the functions $\Phi$and η, the time derivative of the averaged potential takes the form

$$\frac{d\hat{E}(t)}{dt} = \frac{1}{\mid \Omega \mid} \int\limits_{\Omega} \frac{\partial}{\partial t}(S(x,y,t)\eta(x,y,t))dxdy. \tag{8}$$

This expression makes it possible to analyze the rate of change of the solar energy potential and to study the influence of diurnal and seasonal variations in solar irradiance.

To analyze the cumulative energy potential of the territory over the time interval $[t_0, t_1]$, we introduce the integral characteristic

$$\hat{E}_{\text{tot}} = \frac{1}{\mid \Omega \mid} \int\limits_{t_0}^{t_1} \int\limits_{\Omega} S(x,y,t)\eta(x,y,t)dxdydt. \tag{9}$$

The quantity $\hat{E}_{\text{tot}}$ is convenient for comparing the energy resources of different territories over the same time period and can be used as a generalized indicator of solar energy potential.

The proposed model separates fundamental astronomical factors from urban spatio-temporal heterogeneity, allowing independent analysis and flexibility with various data sources.

5. Numerical Implementation of the Model. For the practical application of the proposed model and for obtaining quantitative estimates of solar energy potential, we consider its discretization in space and time.

Let the domain Ω be approximated by a regular grid with spatial steps Δx and Δy. Denote by Ω_h the set of grid nodes belonging to Ω, and introduce the indexing (x_i, y_j), where $i = 1, \ldots, N_x$ and $j = 1, \ldots, N_y$. The elementary area of a grid cell is defined as

$$\Delta A = \Delta x \Delta y. \tag{10}$$

Then the spatial integral in formula (2) is approximated by a quadrature sum:

$$E(t) \approx E_h(t) = \sum_{(i,j)\in\Omega_h} S(x_i, y_j, t)\eta(x_i, y_j, t)\Delta A. \tag{11}$$

In the case of irregular discretization or when a territorial mask is applied, weighting coefficients $w_{ij} \in [0, 1]$ may be used to represent the fraction of the grid cell area belonging to Ω. In this case,

$$E_h(t) = \sum_{(i,j)\in\Omega_h} w_{ij} S(x_i, y_j, t)\eta(x_i, y_j, t)\Delta A. \tag{12}$$

Let the time interval $[t_0, t_1]$ be divided into N_t steps with time step Δt:

$$t_k = t_0 + k\Delta t, k = 0, 1, \ldots, N_t. \tag{13}$$

Then the integral in (3) is approximated by temporal quadrature, for example, using the rectangle method:

$$E_{\text{tot}} \approx E_{\text{tot}}^{(h)} = \sum_{k=0}^{N_t-1} E_h(t_k)\Delta t. \tag{14}$$

If higher accuracy is required, the trapezoidal rule may be employed:

$$E_{\text{tot}}^{(h)} = \sum_{k=0}^{N_t-1} \frac{E_h(t_k) + E_h(t_{k+1})}{2}\Delta t. \tag{15}$$

According to (1), solar radiation at grid nodes is computed as

$$S(x_i, y_j, t_k) = S_0(t_k, \varphi)\,\Phi(x_i, y_j, t_k). \tag{16}$$

Here $S_0(t_k, \varphi)$ is the analytical component determined for a given latitude φ and discrete time instants t_k. The corrective function $\Phi(x_i, y_j, t_k)$ is specified in the form of discrete values obtained from spatio-temporal observational data (e.g., satellite or reanalysis data) or constructed as a parameterized function followed by calibration.

The local efficiency $\eta(x_i, y_j, t_k)$ is determined based on the temperature model (4). In discrete form, it is given by

$$\eta(x_i, y_j, t_k) = \eta_{\text{ref}}\left[1 - \beta\left(T(x_i, y_j, t_k) - T_{\text{ref}}\right)\right]. \tag{17}$$

The temperature field $T(x_i, y_j, t_k)$ may be specified either as a function of $S(x_i, y_j, t_k)$ (parameterized dependence) or as an input field obtained from observational or modeling data. In the general case, this does not change the structure of the numerical implementation of formulas (11)–(15), but only affects the procedure for constructing η. In the present numerical experiments, the temperature field T is defined in degrees Celsius (C°) and specified in a synthetic parametric form as a function of solar radiation, $T = f(S)$, in order to isolate and analyze the influence of thermal effects within the proposed modeling framework. The use of synthetic temperature data ensures full control over model parameters and avoids dependence on external datasets, while preserving the generality of the formulation. The proposed numerical scheme, however, remains directly applicable when realistic temperature fields obtained from observations or atmospheric models are used as input.

The numerical computation of the integral energy potential can be formulated as the following algorithm. Specify the domain Ω, grid Ω_h, weights w_{ij}, and spatial steps Δx, Δy. Specify the temporal grid $\{t_k\}_{k=0}^{N_t}$ with time step Δt. For each t_k, compute the analytical component $S_0(t_k, \varphi)$. Construct $\Phi(x_i, y_j, t_k)$ and compute $S(x_i, y_j, t_k)$ using (16). Construct the temperature field $T(x_i, y_j, t_k)$ and compute $\eta(x_i, y_j, t_k)$ using (17). Compute $E_h(t_k)$ using (11) or (12). Compute the integral characteristic $E_{\text{tot}}^{(h)}$ using (14) or (15).

The accuracy of the approximation is determined by the discretization steps Δx, Δy, and Δt, as well as by the choice of quadrature schemes. Reducing Δx and Δy improves the accuracy of spatial integral approximation, while the use of the trapezoidal rule in time (15) reduces integration error for time-dependent processes.

6. Results of Numerical Modeling. To demonstrate the performance of the model (2)–(4) and to assess the impact of spatio-temporal heterogeneity of solar radiation, numerical experiments were carried out using synthetic input data. The domain $\Omega = [0, 1] \times [0, 1]$ and the time interval $t \in [0, 24]$h were considered. The analytical component of solar irradiance was specified as $S_0(t) = 1000 \sin\left(\frac{\pi t}{24}\right)$, and the full radiation field was defined by the hybrid relation $S(x, y, t) = S_0(t)\Phi(x, y, t)$. Photovoltaic efficiency was modeled using a temperature-dependent relation $\eta(T) = \eta_{\text{ref}}\,[1 - \beta\,(T - T_{\text{ref}})]$, with parameters $\eta_{\text{ref}} = 0.2$, $\beta = 0.004\ {}^\circ C^{-1}$, $T_{\text{ref}} = 25\ {}^\circ C$. The temperature field was parameterized as $T = T_{\text{ref}} + \gamma S$, with $\gamma = 0.02\ {}^\circ C\ m^2/W$. Spatial discretization was performed on a 300×300 grid, while temporal discretization used $N_t = 480$ steps with $\Delta t = 0.05h$.

Table 1.

Results of numerical modeling

Scenario	$\hat{E}_{\text{tot}}$ (Wh/m^2 over 24 h)	$\max \hat{E}(t)$ (W/m^2)	$mean\hat{E}(t)$ (W/m^2)
S1: $\Phi \equiv 1$	2863.78	183.999	119.324
S2: gradient $\alpha = 0.3$	2457.25	158.319	102.386
S2: gradient $\alpha = 0.6$	2039.21	131.679	84.967
S3: oscillating gradient $\alpha = 0.3$	2804.21	183.999	116.842
S3: oscillating gradient $\alpha = 0.6$	2744.20	183.999	114.342

Table 1 presents integral and temporal characteristics of the averaged energy potential $E(t)$ for three scenarios of the corrective function Φ: a homogeneous field $\Phi \equiv 1$ (S1), a spatial gradient $\Phi(x, y, t) = 1 - \alpha\frac{x}{L}$ (S2), and an oscillating gradient$\Phi(x, y, t) = 1 - \alpha\frac{x}{L}\cos\left(\frac{2\pi t}{24}\right)$ (S3). The results show that introducing a spatial gradient leads to a significant reduction in daily energy yield: for $\alpha = 0.3$,

$E_{\text{tot}} = 2457.25\ Wh/m^2$, and for $\alpha = 0.6$, $E_{\text{tot}} = 2039.21\ Wh/m^2$, compared to $E_{\text{tot}} = 2863.78\ Wh/m^2$ in the homogeneous case (S1). Thus, even a simple form of spatial heterogeneity can reduce the integrated energy potential by 14–29% relative to the homogeneous model.

For the oscillating gradient scenario, the reduction in the integrated potential is less pronounced ($E_{\text{tot}} = 2804.21\ Wh/m^2$ for $\alpha = 0.3$ and $E_{\text{tot}} = 2744.20\ Wh/m^2$ for $\alpha = 0.6$), while the peak values $\max E(t)$ coincide with those of the homogeneous case. This behavior is explained by the presence of time instants when $\cos(2\pi t/24) = 0$, causing the field Φ to become close to homogeneous.

Additionally, the effect of the temperature correction in the efficiency model was analyzed. For the homogeneous scenario S1, accounting for $\eta(T)$ reduces the daily energy from $3055.78\ Wh/m^2$ (constant efficiency $\eta = \eta_{\text{ref}}$) to $2863.78\ Wh/m^2$, corresponding to a reduction of 6.28%. Similarly, for the gradient scenario S2 with $\alpha = 0.3$, the reduction amounts to 5.40%. These results confirm the importance of incorporating the temperature factor even in simplified modeling frameworks.

7. Conclusions and future research directions. A mathematical model of the spatio-temporal distribution of solar energy potential in urbanized areas has been developed. Solar radiation is represented as a spatio-temporal field combining an analytical component with a corrective term that accounts for environmental heterogeneity, reducing the assessment of energy potential to a well-defined integral formulation. The model generalizes classical homogeneous approaches and enables the evaluation of both local and integral energy characteristics. The results demonstrate that spatial heterogeneity significantly affects energy estimates: introducing a spatial gradient reduces daily solar energy potential by 14–29%, while spatio-temporal variability primarily decreases mean energy flux without significantly affecting peak values. Additionally, accounting for temperature-dependent photovoltaic efficiency leads to a further reduction of integrated potential by approximately 5–6%. These findings confirm that the proposed model provides a flexible and consistent framework for analyzing solar energy potential in urban environments and for comparing different spatial and operational scenarios. Future research will focus on extending the model toward a more realistic representation of urban systems. This includes incorporating surface orientation and tilt, modeling shading effects and building geometry, integrating real spatio-temporal observational data for calibration, and applying optimization methods for photovoltaic system placement.

Conflicts of interest

The authors declare that they have no conflicts of interest in relation to the current study, including financial, personal, authorship, or any other, that could affect the study, as well as the results reported in this paper.

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Data availability

All data are available, either in numerical or graphical form, in the main text of the manuscript.

Use of artificial intelligence

The authors confirm that they did not use artificial intelligence technologies when creating the current work.

Author contributions

Topylko P.: Supervision, Conceptualization, Validation, Construction of Examples, Writing — review & editing, Dankanych A.: Methodology, Formal analysis, Mathematical modeling, Writing — original draft.

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Топилко П. І., Данканич А. О. Математичне моделювання просторово-часового розподілу сонячного енергетичного потенціалу урбанізованих територій.

У статті розроблено математичну модель просторово-часового розподілу сонячного енергетичного потенціалу урбанізованих територій. Сонячну радіацію подано у вигляді просторово-часової функції, що поєднує аналітичний опис інсоляції та коригувальну складову, яка враховує просторову й часову неоднорідність середовища. Енергетичний потенціал формалізовано як інтегральну характеристику за площею території та часовим інтервалом з урахуванням температурної залежності ефективності фотоелектричного перетворення. Запропоновано чисельну реалізацію моделі та виконано обчислювальні експерименти на синтетичних вхідних даних. Показано, що просторові неоднорідності сонячної радіації можуть зменшувати інтегральний енергетичний потенціал на 14–29% порівняно з однорідним випадком, а врахування температурного чинника призводить до додаткового зниження на 5–6%. Отримані результати підтверджують доцільність просторово-часового підходу до оцінки сонячного енергетичного потенціалу урбанізованих територій.

Ключові слова: математичне моделювання, сонячна енергетика, просторово-часовий розподіл, сонячна радіація, урбанізовані території.

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